Gauss’ Discovery of the Hypergeometric Nature of Physics

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1 Introduction

An ever-present question relentlessly confronts the reader of Gauss’ *Summarische Uebersicht*: What was unique, principally, in Gauss’ approach to the problem of determining the orbit of Ceres, which advantaged him to succeed where all others had failed? This writing does not purport to offer an explicit answer to this query; rather, here will proceed the unraveling of a particular hypothesis, which arose in the struggle to resolve this question. The least which might be offered at this point, however, is that, by way of this present investigation, one should begin to see why the answer to this question remains so persistently elusive. In addition to this, it is hoped that this report might aid in defining the specific quality of the answer of which we are in search. To these ends, we here take up a subject, which, though never explicitly mentioned by Gauss in his published works on astronomical subjects, first presented itself to him in its full magnificence while he struggled with solving one of the most difficult problems in all astronomy.¹ That subject: hypergeometric series.

2 Post Hoc

The first public mention of hypergeometric series by Gauss appeared in his 1799 doctoral dissertation, *Demonstratio nova theorematis*. The first part of this work consisted of a series of penetrating critical analyses and refutations of the previous failed attempts to prove the Fundamental Theorem of Algebra which had been given by some of the idolized contemporary mathematicians. In fact, Gauss’ insight into the impostures of Euler, Lagrange, Bernhard Riemann (1826-1866), in an announcement serving to provide a brief historical background to his 1857 *Beiträge zur Theorie der durch die Gauss’sche Reihe* $F(\alpha, \beta, \gamma; x)$ *darstellbaren Funktionen*, pointed out that it was apparently Gauss’ astronomical researches which led him to the development of his concepts pertaining to hypergeometric series. Riemann also took this same opportunity to set the record straight on matters of precedence: almost all the advances in hypergeometric series attributed to Gauss’ contemporaries were found, by way of papers available after his death, to have been independently achieved by Gauss, not only at earlier dates, but often in more fully developed forms.
and D’ALEMBERT, on this matter, is so incisive that, upon first reading the paper, one is imparted a prescience that there must be some other vantage point from which he is approaching the problem. Once the second half of the paper is reached, this foreknowledge is completely confirmed.

In the second part of the paper, Gauss shares his vantage point with the reader, revealing that his insight was not a matter of a superhuman acumen, but rather was one of method. He, unlike his contemporaries, was willing to seek out, rather than conceal, the physical implications of $\sqrt{-1}$. In doing so, he enabled himself to conceive of a higher domain of transcendental functions which was capable of generating all the relationships found in algebraic functions.

Furthermore, from a comparison of a Cartesian representation of an algebraic function, to the one which Gauss discovers, one can begin to see why Gauss had such an ‘as-if-from-above’ view of everything.

This first discussion of hypergeometric series appears in his refutation of D’ALEMBERT’s alleged proof of the Fundamental Theorem of Algebra. Summarily, D’ALEMBERT puts forth the argument that NEWTON’s method for achieving a numerical solution for roots of an algebraic equation using converging infinite series can be applied not only in cases of finding real roots, but also in cases where the roots take on the form $a + b\sqrt{-1}$. Gauss points out that D’ALEMBERT ignores the possibility that the infinite series, which might emerge in such cases, could be of the hypergeometric form, in which instance the series would become divergent and absolutely useless.

Furthermore, not only did Gauss draw attention to D’ALEMBERT’s lack of consideration of such series, he also takes the opportunity to reprove EULER for similar presumptions. Namely, in a footnote to this section, Gauss reprimands EULER for using hypergeometric series in his calculus textbook with the assumption that such series would converge. By pointing out the fact that the series in question were actually divergent, the validity of EULER’s conclusions were consequently discredited.

Yet, typical of Gauss, his aim was never merely to tear down the work of others, nor to promote the authority of his own work, but instead to restitute a method of investigation that would open the field for others to independently explore. Consequently, one finds in his dissertation an abundance of questions, each of which represent a rich pathway awaiting a daring explorer. In this spirit, at the conclusion of his reprimand of EULER, he indicates a door beckoning to be entered:

This has, as far as I know, been noticed by no one until now. Thus it is exceedingly desirable to clearly and rigorously demonstrate why such series, which initially converge very strongly, then ever weaker

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3 The specific reference is to EULER’s use of infinite series to differentiate transcendental functions. In essence, his methods amount to attempting to reduce transcendental functions to algebraic representations by means of infinite series.
and weaker, and finally diverge more and more, nevertheless yield
nearly the exact sum, only in the event that not too many terms are
taken; and in how far such a sum may, with reliability, be taken as
correct.

- Gauss, *Demonstratio nova theorematis*

Gauss would not again publicly mention hypergeometric series for 13 years,
yet when he did, he would do so in no perfunctory way. On January 30th, 1812,
Gauss presented an essay to the Royal Society of the Sciences at Göttingen
in which he unveiled his work on transcendental functions, demonstrating that
not only could many simple and even higher transcendental functions be repre-
sented by hypergeometric series, but, furthermore, when taking on such a form,
new interrelationships *amongst* seemingly unrelated transcendental functions
abundantly present themselves.

Such discoveries, however, were perhaps not the most intriguing aspect of
his presentation. A more surprising revelation was pointed out by Gauss: in
the methods he had employed in his *Theoria Motus*, 1809, to develop what he
considered to be the most convenient method for solving a problem in orbital
determination, he had already been *tacitly* employing hypergeometric series!

In light of this revelation, the following questions arise: How far along had he
developed his methods at the point at which he authored the *Theoria Motus*?
Could his presentation in the *Theoria Motus* reflect a derivation other than
what he had actually carried out? When does he first begin examining the
astronomical problem in the way he does—might it go back to his determination
of the Ceres orbit? If so, given the nature of the available Ceres observations,
could his method of hypergeometric series have provided a critical difference in
his determination over the attempts of others? Might there have been something
which confronted him in his astronomical investigations, which prompted his
development of such a method? Or, had he developed his method in earlier
investigations, enabling him to apply it to astronomy?

To commence our search for answers to these questions, let us first familiarize
ourselves with the problem that Gauss was confronted with.

3 Kepler Problem, Yet Again

Let us retrace our steps through the problem of the determination of an
orbit from three geocentric positions and their corresponding times. Ironically,
the first question that a child might ask about the celestial object—*How far away
is that*?—is also the first question that must be answered. What means do we
have of measuring such a distance? No matter how high we climb, the object
seems not to change its apparent distance from us. No immediate terrestrial
metrics available to us seem to size up to the problem. Thus, the initial challenge
confronting the inquirer is one of determining what metric is *intrinsic* to the
nature of the action they behold. Fortunately we are not bound by the foot, for
Kepler had proffered us his feat.
With Kepler then, as one contemplates an object of the Solar System, one knows with certainty that that object expresses all the harmonic principles which, as Kepler discovered, bound all the parts into a unity. Though we may not know how far away a planet is, we know that that distance is bounded by an entire orbit, wherein that orbit itself is bounded by the harmonic characteristics of a solar system. From Kepler’s *Astronomia nova*, we know that the distance of a planet to the Sun is a quantum of the action of sweeping out equal areas in equal times. It is further known, from Kepler’s *Harmonice Mundi*, that the two quanta, of the Earth’s distance to the Sun and a planet’s distance to the Sun, are related to each other by the respective major axes or parameters. Consequently, it is only through the knowledge of these harmonic metrics that we are able to ascend to measure the distance of a planet from the Earth.

However great this ascent might be, it is not the only conceptual mountain that must be scaled. Once we obtain distances of the planet to the Earth, there is still much to be accomplished. Working back from a geocentric perspective to a heliocentric one is evident enough as a matter of spherical trigonometry. Once that is achieved, the first two elements, the inclination of the orbital plane to the ecliptic and the longitude of the node, readily present themselves to the resourceful pursuer. After this though, the hasty and the sure-paced alike will be halted in their paths, for once again an age-old problem begs them pay their obeisance and dedicate some solemn moments of their pilgrimage to comprehending the incomprehensible incomprehensibly.

### 3.1 Gauss’ Homage

Gauss provides the reader of his *Theoria Motus* an introduction to this age-old problem in §84 as follows:

> Since it is possible to determine the whole orbit by two radii vectors given in magnitude and position together with one element of the orbit, the *time* also in which the heavenly body moves from one radius vector to another, may be determined... Hence, inversely, it is apparent that two radii vectors given in magnitude and position, together with the time in which the heavenly body describes the intermediate space, determine the whole orbit. But this problem, to be considered among the most important in the theory of the motions of the heavenly bodies, is not so easily solved, since the expression of the time in terms of the elements is transcendental, and, moreover, very complicated. It is so much the more worthy of being carefully investigated...

Taking Gauss’ advice then, it is this problem that we will henceforth dedicate ourselves to resolve, in hopes that we might attain what Gauss regarded as so worthy of our effort. Before we directly take on the “very complicated” challenge, let us first familiarize ourselves with the problem at hand by demonstrating to ourselves that the first part of Gauss’ statement really is possible. That is, let us first take up the problem of determining an orbit beginning from
the case where we have already obtained the magnitude of two of the radii vectors together with one of the elements. After having accomplished this, we shall then attempt to invert the process.

To begin, an example will be most suitable for our purposes. Let us take the case directly from Gauss’ calculation of the orbit of Ceres, the elements of which were presented by von Zach in the December issue of the *Monatliche Correspondenz*.⁴ For the determination, Gauss used the three observations

<table>
<thead>
<tr>
<th>Date</th>
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<th>&quot;</th>
</tr>
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<tbody>
<tr>
<td>Jan. 01</td>
<td>8</td>
<td>43</td>
<td>17.8</td>
</tr>
<tr>
<td>Jan. 21</td>
<td>7</td>
<td>24</td>
<td>2.7</td>
</tr>
<tr>
<td>Feb. 11</td>
<td>6</td>
<td>11</td>
<td>58.2</td>
</tr>
</tbody>
</table>

from Piazzi’s 1801 data. If we take for the calculation the radii vectors from the outer two observations, then we should arrive at a calculated time elapsed of 40.8949120 days.

Let the radii for the first and third observations be \( r = 2.7337947 \), \( r' = 2.7685475 \), and the angle between them be \( \theta = 8°49'24''09 \) or, expressed in parts of the radius, \( \theta = 0.1539967 \). Let us take the semi-parameter as our given element, namely \( p = 2.7451305 \).

If one expresses the radius in terms of the true anomaly the following relationship is found:

\[
r = \frac{p}{1 + e \cos v}
\]

where \( e \) and \( v \) denote the eccentricity and true anomaly, respectively.⁵

Separating our known quantities, we will have the two relations:

\[
e \cos v = \frac{p}{r} - 1 = 0.0041465
\]
\[
e \cos v' = \frac{p}{r'} - 1 = -0.0084582
\]

What can we draw from these relationships? For one, since we know that the eccentricity must be positive, the signs of these cosines, together with the presumption that the observations occur at successive times, give us insight into the qualitative positions of these observations with respect to the perihelion. Namely, since the cosine is positive, and the distance to the Sun is less than the semi-parameter, for the first observation, we can conclude that the observation must have occurred when the planet was nearer to the perihelion than the aphelion. Further, since the next observation, which is only a little more than \( 8° \) beyond the first, has a negative cosine, and has a distance from the Sun greater than the semi-parameter, the planet must have gone past the quadrant in the direction of the aphelion. Thus, we already know something qualitative about these positions with respect to the entire orbit.

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⁴Gauss provided von Zach four sets of elements, of which the data given here is calculated from the third set.

⁵The notation adopted here coincides with that used by Gauss in the *Theoria Motus*, and are developed there in §§1-8.
We still do not quantitatively know the value of either $e$ or the true anomalies, so we must look to see if these relationships formed from our equations might be derived from something we do know. Re-examining the equations,

$$e \cos v = 0.0041465 \text{ and } e \cos v' = -0.0084582,$$

one may notice that it is possible to arrange these relationships on a circle with radius $= e$ and angles $= v, = v'$.

If we set $e \cos v = x$ and $e \cos v' = y$, then one can prove the relation: $^6$

$$e \sin (v' - v) = \sqrt{x^2 + y^2 - 2xy \cos (v' - v)}$$

$^6$The elegance of this relationship demands that readers discover it for themselves.
Hereafter the eccentricity will be easily calculated, namely

\[ e = \sqrt{x^2 + y^2 - 2xy \cos \theta} = 0.819604, \]

where \( \theta = v' - v \).

Now, knowing the eccentricity, our equations will yield us values of the respective true anomalies for those positions, \( v = 87^\circ6'0''.18 \) and \( v' = 95^\circ55'24''.28 \), or \( v = 1.5201827 \) and \( v' = 1.6741794 \), in parts of the radius. The semi-parameter can also be converted into the semi-major axis, \( a \), since 

\[ \cos \phi = \sqrt{1 - e^2}, \]

Thus, \( a = 2.7636956 \).

3.2 Inversion

That seemed to be not as difficult as one might have expected. Yet, if anyone remembers Chapter 60 of Kepler’s *Astronomia nova*, though calculating the mean anomaly from the eccentric anomaly presented little difficulty, inverting that calculation proved an insuperable task. The question now: Can we invert the process which was just completed? That is, given the two radii

\[ \text{time elapsed (in years)} = \frac{\Delta M}{2\pi a^2} \]

Multiplying this by the number of days in a sidereal year, 365.2563835, we obtain 40.8949302 days elapsed. If we compare this calculated time elapsed to the above given observed time elapsed, it will provide us a means of checking the accuracy of our calculated elements. Above we found that the observed time elapsed was 40.8949120. Thus the difference between our calculated time elapsed and our observed time is 0.0000112 days, or a little less than a second - quite an acceptable degree of accuracy!

Before the time of the 17th-century invention of Christian Huyghens’ pendulum clock, the error accumulated by clocks in a day was measured in minutes. With Huyghens’ first implementation, this error was reduced to about one minute per day, and by the end of his life was brought down to ten seconds. By the time of Gauss’ day, clocks lost around a second in accuracy over the course of a week. Thus, one second accumulated over the course of forty days (which is what our computed error amounts to) falls well within the limits of instrumentation of those days.
vectors, the angle between them, and the time elapsed, can we derive a value for the semi-parameter, eccentricity, or position of the perihelion?

If we attempted a simple stepwise inversion, what would be our first step? In the opposite direction, the last step we made involved converting the mean motion into a number of days, such that our first step in this direction should be converting our number of days into a mean motion. Yet if we look at the equation,

$$\frac{2\pi \cdot \text{time elapsed (in years)}}{a^2} = \Delta M$$

one is confronted with the problem that the value of the semi-axis major is as yet unknown. Further, knowing the semi-axis major involves knowledge of the parameter and the eccentricity, neither of which we have. Thus, one must try to express the major-axis in terms of our known values.

Supposing that this is possible, we are faced with yet another problem: once we obtain the mean motion, we are wont to proceed to the eccentric anomalies. Here is where the Kepler Problem arises, yet now it takes on an added difficulty. Since we do not know either of the mean positions independently, but only their difference, we must rather take on the more difficult task of finding the difference in the eccentric anomalies.

$$\Delta M = (E' - e \sin E') - (E - e \sin E)$$

Now the reader might, appropriately, become wary of taking this path, for even in the original Kepler Problem, of finding a single eccentric anomaly from a single mean anomaly, this could not be solved explicitly, but could only be approximated by an iterative process. Now, instead of merely solving the Kepler Problem for a single time, we must pursue a solution to the problem for two times, both of which are unknown to us, although we know their difference.

Here something quite profound begins to suggest itself as underlying the problem being confronted. For at this moment, the difference between a simple geometric elliptical motion and the efficient principled action of the physical elliptical motion found in the motion of planets presents itself as seemingly irreconcilable. Though the expression of this physical action as a whole may clothe itself in what many consider to be a simple geometric figure, the temporal unfolding of that action wholly transcends any attempted identification of it with its empirical appearance. Ironically, Kepler was successful in identifying the nature of such action (and even communicating it to his fellow man), as his *Astronomia nova* attests, yet, fundamentally his comprehension of it could still only be stated in the form of a paradox.
One might object, “But do we not know the laws of planetary motion?” “Have these not been precisely defined?” “How could Kepler claim to discover a ‘principle’, yet not be capable of expressing it mathematically?” “How can one be considered to know something, if all they really know is that they do not know it?”

For now, we will leave it to the reader to attempt this pathway on their own (though we will be returning to this trail when we directly take up Gauss’ approach).

3.3 Another Route

At this point we will make an attempt to circumvent these difficulties by investigating the applicability of an approximation technique. The reader should already have encountered one approximation technique for this problem, which will not be treated here. Let us introduce another approach for pedagogical purposes.

After Kepler had devastatingly demonstrated the futility of adhering to mere geometric modeling for comprehending the motions of the planets in his *Astronomia nova*, he proceeded to introduce a new mode of investigation to astronomy: hypothesizing physical causality. With this concept of the knowability of physical causality, he would arrive at a new hypothesis that would transform all astronomical investigations thereafter. Yet he admitted, even as early as his *Mysterium cosmographicum*, that the singular hypothesis which he arrived at, that the planetary motions locate their source and ordering in the Sun, was not entirely new, but rather was a reincarnation of Pythagorean knowledge.

Kepler first begins to elaborate upon this hypothesis for the reader of his *Astronomia nova* in the thirty-second chapter. In order to establish the existence of such a causal relation, he investigates the interaction of a planet’s angular motion, both about the center of the orbit and about the Sun, with the planet’s distance to the Sun. By means of this investigation he is able to demonstrate that the effect of the planet’s speeding up and slowing down along different parts of its orbit, which previously had only been treated geometrically, could be the result of a physical principle. Yet at that point, he still did not establish what figure the motion resulting traversed, but rather only how the motion changed.

These differential relations, which he established in Chapter 32, specified that the angle about the center changes inversely proportional to the distance from the Sun, and, consequently, that the angle about the Sun changes in an inverse proportion to the square of the distance from the Sun. In the subsequent chapters of the book (33-38), Kepler commences the search for what must be the nature of the planet and Sun, in order that such relations could exist. After this, much of the remainder of the *Astronomia nova* is directed toward seeking

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8To deal with this problem, Gauss presented an approximation method in §7 of his *Summarische Uebersicht*. There he used an integral approximation technique, gathered from Cotes, to derive a suitable value for the area swept out. Once the area was known, it could be divided by the time in order to establish an approximate value for the semi-parameter.
out what motion might satisfy such differential relationships—that is, what figure these differential relations generate.\(^9\)

Although Kepler’s results had been extracted from his *Astronomia nova* and reduced to the form of “Laws” (i.e., elliptical motion and equal area, equal time), these differential relations were still underlying the analytic representations that had become prevalent by Gauss’ day.\(^10\) Namely, it can be shown that the second of these relations, that between the angle about the Sun with the distance from the Sun, finds itself expressed as

\[
\frac{dv}{dM} = \frac{a \cdot b}{r^2}
\]

where \(v\) is the true anomaly, \(M\) the corresponding mean anomaly, \(a\) and \(b\) the semi-axes major and minor, respectively, and \(r\) the distance from the Sun. It is this very relationship, as Kepler knew it, which we will here attempt to use in order to solve our problem.

From what was learned about the differential analysis found in Gauss’ application of orders in his analysis of approximation techniques, we can know that, in the infinitesimal, this relationship is a precise representation of the nature of the action occurring. That is, if the difference between the two times of observation is considered to be infinitesimally small, this relation will hold true. As a warning, though, when we proceed to the finite, these relationships can only provide an approximation of the truth, and are thus susceptible to significant error.

Let us see what we can make of it for our purposes now.

Were the time elapsed between our two positions infinitely small, then the angle between them, which we know, would amount to \(dv\). Furthermore, the product of the two radii would then differ infinitely little from the square of either of the radii. The last part to be developed would be \(dM\). For this value we know

\[
\frac{\text{Mean Anomaly}}{2\pi} = \frac{\text{Time Elapsed}}{\text{Length of Period}}
\]

or

\[
\frac{M}{2\pi} = \frac{t}{a^2}.
\]

Differentiating (i.e., taking our time elapsed to be infinitely small) we have

\[
dM = 2\pi \frac{dt}{a^2}.
\]

\(^9\)As best I can gather, in matters of precedence, the *Astronomia nova* provides the first determination, investigation, and solution of what is in modern terms called a differential equation. Thus, it would be most beneficial to compare the method there employed by Kepler to the methods used later by Bernoulli and Leibniz respecting the catenary curve, as well as those of the young Euler respecting the curve elastica.

\(^{10}\)Should an adventurous reader want to discover this for themselves, an investigation into the differential relations amongst the equations of planetary elliptical motion, as found in Gauss’ *Theoria Motus*, §8, will not prove unfruitful.
Substituting this value for \( dM \) into our equation we arrive at

\[
\frac{dv}{2\pi \cdot dt} = \frac{\sqrt{a \cos^2 \phi}}{r^2} = \frac{\sqrt{p}}{r^2}
\]

since \( b = a \cos \phi \). From this we obtain a value for the semi-parameter

\[
\sqrt{p} = \frac{dv}{2\pi \cdot dt} r^2.
\]

If we extrapolate this equation into the finite, will it provide an adequate approximation for the semi-parameter? One should be able to analyze this with the tools Gauss demonstrated in his *Summarische Uebersicht*, where he applied infinitesimal analysis to determine the accuracy of such approximations.

Just to check with some examples, here would be the results were we to use the three observations Gauss chose (again using the comparison of observed times to calculated times elapsed as a metric for our error):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Calculated Time (days)</th>
<th>Calculated Time (seconds)</th>
</tr>
</thead>
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<tr>
<td>Jan 1st to Jan 21st</td>
<td>2.7451371</td>
<td>19.9449538</td>
</tr>
<tr>
<td>Jan 21st to Feb 11th</td>
<td>2.7451019</td>
<td>20.9506020</td>
</tr>
<tr>
<td>Jan 1st to Feb 11th</td>
<td>2.7450844</td>
<td>40.8952763</td>
</tr>
</tbody>
</table>

Even from this short table alone one can see that the rate at which error accumulates as the observations become further apart is rather discouraging for using this technique in general. In fact, for angles this small the first method that Gauss provided would be sensibly adequate. Regardless of accuracy, however, there is still something unsatisfying about either of these two methods: they both avoid the question of the *substantial action* which is actually occurring. Though these methods might employ other principled relations which exist in the planetary motion, neither one attempts to develop the exact relation which exists between the change in time and the change in the eccentric anomaly. In both these techniques, one proceeds immediately to an approximation without ever investigating the uniqueness of the orbital relationships themselves. Thus, they might answer a question, but an important stumbling block is left unturned.

Let us rendezvous with Gauss and take up again the question of when he himself first began to investigate this problem in a more principled way. To this end, perhaps his correspondence might provide some insight.

4 The Dialogue

In Gauss’ presentation of his method for the determination of planetary orbits, which he sent to Olbers, when he covers the topic of how to determine the elements of a planet given two radii vectors in an orbit, he apparently makes very little deviation from what had been standard procedure up until
that time.\textsuperscript{11} OLBERS, seemingly eager to provoke the young Gauss into developing new methods in this aspect of determining orbits, writes, in his letter of September 11, 1802:

XII. Your methods to determine the orbit from \(v, v'', r, r''\) and especially 3., are very fine, clever, and useable. Nevertheless, the following thereby occurred to me. It is well known that LAMBERT has given a very elegant series to find the time in which the elliptical sector is traversed from the chord, both of the \textit{radius vectors}, and the major axis. He comes upon his series synthetically. LAGRANGE had also proved it analytically in the \textit{Memoirs of the Berlin Academy}, 1778. Thus, there must also be a series inverted in the time, chord, and both the distances, to find the major axis. It will only depend upon whether this series is convenient to calculate, and is sufficiently convergent. If the latter were not the case for a series for \(a\), then it could perhaps be so for \(\frac{1}{a}\). I have not yet tried it, but I fear that these series will only become strongly convergent, if \(a\) is very large compared to \(r\) and \(r''\). Otherwise it were good, I think, if \(a\) were found by a series directly, and likewise all the remaining elements without effort.

Though Gauss neglects to respond to this suggestion by OLBERS in his next letter, as he is quite occupied with answering OLBERS' sundry other questions, it certainly must have caught his eye. In the following letter of September 21, 1802, Gauss gives him reply:

XII. Your clever suggestion to represent the semi-major axis (or any other element) by means of a series, I like very much, and I will consider it further in the future, although I fear, as you do, that the same would only be advantageous for practical use in special cases. For practical use I find your communicated type of procedure [\textit{Verfassungsart}] for determining the parameter by approximation, to be the more advantageous the more I make use of it. I now put it in the following form:

\[
\begin{align*}
\theta'' - \theta &= \delta \\
\text{\(\theta\)'s mean motion in tempore a \(\tau\) usque ad \(\tau''\) = \(M\)} \\
\frac{r''}{r} &= \tan(45^\circ \pm \phi) \quad \text{Semi-parameter=}\(p\)
\end{align*}
\]

Then, as long as \(\delta\) is not too great,

\[
\sqrt{p} = \frac{\delta r''}{3M \cos 2\varphi} \left(1 + 2 \left[\frac{\cos 2\varphi \cos \frac{1}{2}\delta}{\cos \varphi \left(1 - \frac{2 \sin \frac{1}{4}\delta^2}{p \cos \varphi \sqrt{r'' \cos \varphi}}\right)}\right]^2\right)
\]

\textsuperscript{11}See write up on SU §7 by Liona.
is always extremely exact, and

\[ p = \left( \frac{rr''\delta}{M} \right)^2 \sqrt{\left( \cos 2\varphi^2 \left( \frac{\cos \frac{1}{2} \delta}{\cos \varphi} \left( 1 - \frac{2 \sin \frac{1}{4} \delta^2}{p \cos \varphi} \sqrt{rr'' \cos \varphi} \right) \right)^8} \]

is almost just as exact, but more convenient to calculate.

I wish that you yourself would make an attempt, in order to see for yourself how conveniently and quickly one can calculate according to this formula. With my latest calculations of Pallas’ orbit, where \( \delta = 24\frac{1}{2} \) degrees, I had used the other method, since I already knew the elements nearly, and made two false positions for the aphelion, whereby \( \log p \) would be = 0.4158666. Out of curiosity, I have now derived \( p \) according to the same, in order to see how exactly these formulas of approximation would give \( p \). The first gave \( \log p = 0.4158612 \); the second = 0.4158535.

Gauss clearly seems excited to share with Olbers these variations on previous formulae, especially since they present a new degree of marriage between ease of calculation and degree of accuracy, two virtues held in quite high esteem for the hand calculator. Yet, as one can see, Olbers’ suggestion is still left to be investigated.

Olbers, before trying out Gauss’ new formulae for himself, sends Gauss this response in an October 10th, 1802 letter:

The more I consider your method to find the parameter from \( v, v'', r, r'' \), the more easy and attractive I find it to be; you have especially made the calculation far more convenient still in your last letter. — No, a series for \( a \) or \( k \) etc. can certainly never be as useful for the calculation. — As soon as I have the time, as well as the occasion, I will attempt to calculate an example according to your method, purely for my own practice; for at the beginning it may go for me similarly as with Scanderberg’s sword.\(^{12}\) Truly, the certainty and precision with which you now calculate according to your formula is very difficult for me. I miscalculate very easily, especially in small matters, which do not immediately stand out in the result.

\(^{12}\)A well-known saying of the time, which goes something like: “Scanderbeg’s sword must have Scanderbeg’s arm,” alluding to the story of the 15th century Albanian prince Gjergj Kastrioti, or, as known to the Turks, “I slander-beq” (Prince Alexander), who sent a scimitar as a gift to his enemy, the Turkish Emperor Mahomet. However, the Emperor could not so much as lift the sword, and furiously sent it back, viewing it as an attempt to inspire fear. The prince calmly responded that he had merely sent his own sword, but not the arm which had wielded it in victorious war against the Turks. (Martim de Albuquerque’s Notes and Queries, 1853.) [TAD]
Thus Olbers humbly concedes the superiority of Gauss’ insight into such computational practices, and he himself goes back against his original suggestion, assailing the usefulness of applying series to solve the problem. Such was the state of this problem left for a number of years, in so far as it was discussed between Gauss and Olbers at least. Eventually the entire subject of this initial determination of an orbit is found absent in their discussion, as it is displaced by more pressing matters, such as the problem of perturbations, Gauss’ appointment to a full time position in either Göttingen or St. Petersburg, and Gauss’ growing excitement in making his own observations, especially geodetic ones.

In an October 29, 1805 letter, Gauss makes a brief note that he is preparing to undertake an elaboration of his work on determining orbits, though his larger preoccupations at the time are considerations of perturbations and arithmetical matters. Other than this indication, he provides no clarification as to whether he is pursuing anything afresh in the field of orbital determinations, or merely refining what he had presented Olbers before.

After a long silence on the matter, on February 3, 1806, Gauss communicates to Olbers that he has had the opportunity to reconsider the task of determining an orbit, and it has yielded him some valuable fruit:

In this year I have diligently worked on my method to determine planetary orbits; although as yet not as much [has been devoted] to the elaboration, than to the greater refinement of individual aspects of the same. Much, I believe, is to my good fortune and in the least [my determination] has obtained an entirely different form than previously, yet I would have but little joy about this, as well as with all the work, did I not have the hope of writing to you. The principle improvements pertain to the problem of determining the planetary elements from two heliocentric positions in the orbit, together with the distances from the Sun. Since you have always received my communications in these labors so well and it may also be desired, thence will I here at least transcribe the results for you, since I am not calm enough for a coherent dissertation.

Gauss proceeds to provide Olbers his results. Although the practical results in his transcription to Olbers coincide with what he presents in his Theoria Motus, the discovered relations to hypergeometric series and the subsequent development of them into a continued fraction he wholly fails to mention. He does, however, indicate what he considers to be entirely new observations about elliptical motion, which will be useful to us as we try to reconstruct his discovery.

Olbers does not have an opportunity to respond to Gauss for over two months, and when he does, on April 29th, 1806, this newly found solution of Gauss’ does not occupy much of his attention:

Your formulas, to determine the elements of a planet from two heliocentric positions in the orbit and the distances, have me very delighted. They are for the greater part new, beautiful and convenient.
To our disappointment, Olbers never develops the same inquisitiveness toward this insight of Gauss' as he did with the *Summarische Uebersicht*, the latter of which found a great deal of elaboration in the dialogue between the two in response to Olbers' tireless scrutiny.

Thus, it seems we are, so far, left to our own speculation. However, we have here found some partial answers to some of our questions, namely:

1. Between October 1805 and February 1806, he seems to have first developed his unique solution to the problem at hand.

2. If this is true, then it would follow that this method was not the one employed in the determination of the Ceres orbit.

3. Olbers had encouraged him to examine the problem afresh, but Gauss did not immediately take this advice, although he may have been reminded of it when he began to re-examine his method of determination.

At the same time, although we may have answered some questions, we are also left with some new ones: Was his approach entirely new, or did he take the advice to explore for an inversion of Lambert's and Lagrange's solutions? Seeing as, in the presentation found in the correspondence, he fails to mention series or continuous fractions, has he perhaps not yet developed his understanding of hypergeometric series at this point?

With that much as an historical introduction, let us now proceed to how Gauss approached this problem.

## 5 The Setup

It is time to get to work. This section of the present paper, although accomplishing everything Gauss presents in §88 of the *Theoria Motus*, is given from the standpoint of how the equations provided in that section might have been geometrically developed. We leave it to the perspicacity of the reader, in comparing this presentation to §88, to make judgment as to which approach is the more plausible development of Gauss' results.\(^{13}\)

For the sake of brevity, we will here at the outset adopt some of the abbreviations that Gauss likewise adopted in the *Theoria Motus*. Let

\[ v' - v = 2f \]
\[ E' - E = 2g \]
\[ E' + E = 2G \]

---

\(^{13}\)The author certainly does not exclude a third possibility: that Gauss' true approach might deviate from both these presentations; for in many instances in his early work one finds that the effortlessness with which the majority of his results were derived take their origin in his understanding of the complex domain, though he notoriously avoids clothing his presentations in these methods until his 1832 *Second Treatise on Biquadratic Residues*. 

15
where \( v, v' \) are the true anomalies, and \( E, E' \) the eccentric anomalies at the times \( t, t' \), respectively. We trust the development will fully disclose the usefulness of these abbreviations.

We now return to our inverse problem. From Kepler we know \( M = E - e \sin E \). Now, however, we will be examining the difference between two mean anomalies, or, more simply put, the mean motion. Stating this in terms of the corresponding eccentric anomalies, our equation thus becomes

\[
M' - M = E' - e \sin E' - (E - e \sin E)
\]

Physically this translates into the sector which is the difference between the two elliptical sectors swept out by the planet. Examining this geometrically, we can proportionally place this in the eccentric circle,\(^{14}\) taking the radius as unity. By doing so, one will find that the area of the difference between the two sectors, now in the eccentric circle, takes on the equivalent expression

\[
M' - M = 2g - 2e \sin g \cos G.
\]

\(^{14}\)That is, the circle circumscribing the ellipse.
This will undergo a further transformation, but first we must make an observation about the relationship between the difference of the true anomalies and that of the eccentric anomalies. This relationship can be acquired by way of comparing two different expressions of the chord between the two positions on the ellipse.

First, the chord, which we will denote by \(c\), can be expressed by the Pythagorean relationship involving the eccentric anomalies

\[
(a \cos E' - a \cos E)^2 + (b \sin E' - b \sin E)^2 = c^2.
\]

Next, the chord can also be expressed through the generalized Pythagorean relationship involving the difference of the true anomalies

\[
r^2 + r'^2 - 2rr' \cos 2f = c^2.
\]

Thus one has the identity

\[
r^2 + r'^2 - 2rr' \cos 2f = (a \cos E' - a \cos E)^2 + (b \sin E' - b \sin E)^2 \]
Though tedious, it is not hopeless to attempt to carry out the reduction of this equation. To outline the reduction, on the right side, $b^2$ should be replaced by its equivalent, $a^2 \cdot (1 - e^2)$, where $e$ is the eccentricity. This will then help to consolidate some of the trigonometric terms appearing on the right side. On the left hand side it is useful to substitute, instead of $\cos 2f$, the identical $2 \cos^2 f - 1$. By doing so, the term $2rr'$ will emerge, which can then be combined with $r^2 + r'^2$ to form $(r + r')^2$. Now, we are well on our way, but first our path must diverge to another not so apparent geometrical relationship, which, once included, will bring this reduction to arrive at a very elegant truth.

Now that we have achieved the consolidation $(r + r')^2$ in our reduction, our next major stride will come by expressing $r + r'$ in terms of the difference in the eccentric anomaly. One should recall from KEPLER’s original construction of the ellipse that the length of the radius was actually constructed from the diametral distance.  

By aid of the diagram, one should be able to conclude that the excess of the one radius over $a - a \cdot e \cdot \cos g \cos G$ is exactly equal to the defect of the other

---

15See KEPLER’s *Astronomia nova*, chapters 56-60.
radius below \( a - a \cdot e \cdot \cos g \cos G \). Thus results

\[
    r + r' = 2a - 2a \cdot e \cdot \cos g \cos G
\]

Taking this result back into our reduction, multiplying out its square, and subtracting the result from both sides, the right-hand side of the equation should consolidate into \(-4(a(cos g - e \cdot \cos G))^2\), yielding the result

\[
    rr' \cos^2 f = (a(cos g - e \cdot \cos G))^2
\]

or

\[
    \sqrt{rr'} \cos f = a(cos g - e \cdot \cos G).
\]

It is this relationship that we will then bring back into our investigation, but first, let us take a short diversion from our main path to explore a most intriguing and beautiful relationship, only a single side of which we have just now begun to survey.

5.1 A Small Diversion of Great Proportions

We now take a moment to examine the triangle between the Sun and two positions on the orbit. How can we express its area? One expression, and perhaps the simplest, is that comprised of the difference in the true anomalies together with the radii; namely,

\[
    \frac{1}{2}rr' \sin 2f.
\]

Can this triangle’s area also be expressed in terms of the eccentric anomalies? If we remember back to the relationship that every ellipse has to its eccentric circle, we might recollect that the proportion between the triangle in the ellipse to the triangle projected up to the eccentric circle is equal to the proportion between the ellipse’s minor axis to its major axis (which is equal to the diameter of the eccentric circle), or \( \cos \varphi : 1 \).
Thus, if we can find an expression for the triangle in the eccentric circle, then by reducing it by a factor of $\cos \varphi$ we will obtain another expression for the triangle in the ellipse.

If we consider the chord between the two positions on the eccentric circle to be the base of the triangle, which can be easily seen to be $2 \sin g$, then we need only find the height of the triangle. Without much difficulty we can conclude that the height of the triangle turns out to be $a \cos g - a \cdot e \cos G$. 
Therefore another expression for the area of the triangle in the ellipse is

\[ \cos \varphi (a \sin g) (a \cos g - a \cdot e \cos G) \]

Combining these two results, we have

\[ \frac{1}{2} r r' \sin 2f = \cos \varphi (a \sin g) (a \cos g - a \cdot e \cos G) \]

Does any of this look familiar? How about if it is rearranged like so:

\[ (\sqrt{rr'} \sin f)(\sqrt{rr'} \cos f) = (b \sin g)(a (\cos g - e \cos G)) \]

Now if we apply the identity which we found immediately before our diversion, we find the relationship between the sines of the true and eccentric anomalies to be

\[ \sqrt{rr'} \sin f = b \sin g \]

It seems we literally were seeing only one side before! Yet, what does this mean geometrically? Where does one find the geometric mean between the radii in this construction?
This being a diversion, the author leaves these questions to be explored by the reader on their own. However, it might be of value here to note that these relationships between the sines and cosines of the differences of the true and eccentric anomalies are precisely the ones which Gauss lays claim to be first discovered by him.

5.2 Uphill

Coming back to the main path, remember that we just identified the relationship, \( \sqrt{rr'} \cos f = a(\cos g - e \cdot \cos G) \), and have now to apply it in our derivation. We had left off with the equation for the mean motion

\[
M' - M = 2g - 2e \sin g \cos G.
\]

Substituting into this equation

\[
-e \cos G = \sqrt{rr'} a \cos f - \cos g
\]

we obtain, after reduction,

\[
M' - M = 2g - \sin 2g + 2 \cos f \sin g \frac{\sqrt{rr'}}{a}.
\]

At this point, it may seem as though we have been taking individual geometric steps, but that we are losing the geometry of the whole. However, if we take a moment to decode what we have arrived at, it will be found that this too has a very intuitive meaning.

For example, it should be obvious that \(2g\) is still the sector in the mean circle (i.e., with unit radius), which is swept out from from the center over time \(t\) to \(t'\).
Next, one might observe that \( \sin 2g \), taken in this circle, is equivalent to the area of the triangle the arc of this sector. Moreover, it is clear that \( 2g - \sin 2g \) is identical to the circular segment created by the chord between the two positions on the eccentric circle with unit radius.

Finally, with not too much effort, it should be evident, since the mean motion
consists of the sector taken from the eccentric point\textsuperscript{16} on the apsides, that the remaining part of the equation, \(2 \cos f \sin g \sqrt{\frac{r}{a^2}}\), must be equal to the area of the triangle between the two positions and the eccentric point.

Also make sure not to forget that all these areas are proportional to the physical areas swept out by the actual motion of the planet by the ratio of 1 : \(a \cdot b\).

The construct here will form the basis for Gauss’ solution to our problem, but there are still further algebraic transformations that must be undertaken before this becomes directly applicable to the problem before us.

\textsuperscript{16}That is, the position demarcated by the Sun in our diagrams.
5.3 The Final Ascent

The author must forewarn the reader that although these final steps involve further reductions requiring algebraic substitutions, etc., we have already achieved the essential form which becomes the basis for Gauss’ application of hypergeometric series. Thus, it is important to remember that despite the seemingly anti-intuitive clothing this latter step assumes, the final representation will not fail to express what has thus far already been attained.

If one takes a moment to reflect on the nature of the problem we are undertaking to resolve, the matter of where one must proceed next should begin to reveal itself. In this manner, if we consider what has been constructed thus far, we still possess one central problem to clear up. To this end, we must keep in mind the fundamental purpose of mathematics itself: to bring what is immediately indeterminate or unknown in magnitude into a unique knowable relationship to what is already known. Or, as KEPLER states it in his Harmonice Mundi, “...to know is to measure by a known measure.” In terms of practice, this translates into reducing our equation into an expression of a single unknown in terms of what we have already numbered.

Hearkening back to the original statement of the problem: “it is apparent that two radii vectors given in magnitude and position, together with the time in which the heavenly body describes the intermediate space, determine the whole orbit.” Though we will not be immediately proceeding from these known magnitudes to one of the elements of the orbit, we are attempting to bring into knowability a presently indeterminate magnitude (i.e., the difference in the eccentric anomaly), which will enable us to directly proceed to the calculation of the orbital elements. Therefore, the final task before us is to eliminate the remaining unknowns in our equation, such that all that remains will be our known magnitudes and this single, isolated unknown.

Perusing our equation,

$$\Delta M = 2g - \sin 2g + 2 \cos f \sin g \sqrt{rr'} a,$$

there are two immediate unknowns, which actually reduce to one and the same. On the left-hand side of our equation, we still do not know the mean motion, but only the terrestrial motion, or the change undergone measured in Earth time. On the right-hand side, we also are still dependent on the value of the semi-axis major, which is one of the elements we wish to acquire. For the former, one shall recall that KEPLER had discovered a very important, harmonic commensurability between the mean motion of a planet and the Earth’s measure of temporal motion. This was the three-halves law:

$$\frac{\text{Mean Motion}}{2\pi} = \frac{t' - t}{a^{3/2}}$$

or, for our purposes,

$$\text{Mean Motion} = \frac{2\pi(t' - t)}{a^{3/2}}.$$
Hence, on both the left and right side of our equation, we have yet to resolve the value of the semi-axis major. Let us take this up immediately.

Earlier we had derived the relationship \( r + r' = 2a - 2a \cdot e \cos g \cos G \), which, if we further express \( \cos G \) by our other relation \( \sqrt{rr'} \cos f = (\cos g - e \cos G) a \), will become

\[
r + r' = 2a \sin^2 g + 2\sqrt{rr'} \cos f \cos g.
\]

Therefore

\[
a = \frac{r + r' - 2\sqrt{rr'} \cos f \cos g}{2 \sin^2 g}
\]

which provides us an expression for \( a \) involving only known quantities and the single unknown \( g \). As is probably apparent at a glance, bringing this value of \( a \) into our equation is going to be messy, which might require some cleaning up.

As a precaution, Gauss suggests first replacing the \( \cos g \) by its identical form \( 1 - 2 \sin^2 \frac{1}{2}g \) and consolidating the numerator into a product, which will make it simpler to separate the terms once we raise it to the \( \frac{3}{2} \)-power.

\[
a = \frac{2 \left( r + r' - \frac{1}{2} + \sin^2 \frac{1}{2}g \right) \sqrt{rr'} \cos f}{\sin^2 g}
\]

Finally, substituting this value for \( a \) into our equation we should, if one brings all magnitudes involving the unknown \( g \) to the right hand side, obtain the final result

\[
\frac{2\pi(t' - t)}{2^\frac{3}{2} \cos^\frac{3}{2} f (rr')^\frac{3}{4}} = \left( \frac{r + r'}{4\sqrt{rr'} \cos f} - \frac{1}{2} + \sin^3 \frac{1}{2}g \right)^\frac{3}{2} + \left( \frac{r + r'}{4\sqrt{rr'} \cos f} - \frac{1}{2} + \sin^3 \frac{1}{2}g \right)^\frac{3}{2} \frac{(2g - \sin 2g)}{\sin^3 g},
\]

albeit a messy one. Gauss recommends, for the sake of brevity, consolidating all the known magnitudes into the symbols \( l \) and \( m \), denoting

\[
l = \frac{r + r'}{4\sqrt{rr'} \cos f} - \frac{1}{2}
\]

and

\[
m = \frac{2\pi(t' - t)}{2^\frac{3}{2} \cos^\frac{3}{2} f (rr')^\frac{3}{4}}.
\]

Doing so arranges our equation in a form so that we can perceive more clearly the particular relationship the unknown magnitude forms with the known. Thus we obtain

\[
m = (l + \sin^2 \frac{1}{2}g)^\frac{3}{2} + (l + \sin^2 \frac{1}{2}g)^\frac{3}{2} \left( \frac{2g - \sin 2g}{\sin^3 g} \right),
\]

which corresponds to Gauss' equation [12] in §88 of the Theoria Motus,\textsuperscript{17} and also to equation I. in his correspondence to OLBERS. He maintains in his correspondence that this equation is “entirely new.”

\textsuperscript{17}In the Theoria Motus, he also develops a similar equation corresponding to instances where the heliocentric motion is between 180° and 360°, which is denoted by [12*].
The attentive reader might have noticed in their derivation that this equation still maintains its original proportionality, but rather than needing to state this in our own terms, Gauss himself is so inspired by these results that he cannot resist pointing out the physical significance himself. In §95, after showing how the orbital elements can be derived from the solution to this equation, he proclaims

This remark is of the greatest importance, and elucidates in a beautiful manner both equations 12, 12*: for it is apparent from this, that in equation 12 the parts $m$, $(l + x)^{\frac{1}{2}}$, $X(l + x)^{\frac{1}{4}}$, etc., are respectively proportional to the area of the sector (between the radii vectores and the elliptic arc), the area of the triangle (between the radii vectores and the chord), the area of the segment (between the arc and the chord), because the first area is evidently equal to the sum...of the other two.\(^{18}\)

We have obtained our equation now, expressing our unknown magnitude $g$ by bringing it into a unique relationship involving only what we know. However, the question remains, have we actually measured $g$? That is, does this equation provide us a commensurable relationship to our unknown? In fact, a crucial incommensurability lies embedded in this explicitly transcendental equation. It seems then, we are thus well on our way to comprehending the incomprehensible incomprehensibly.

6 Finding Our Transcendental Orbit

Gauss could well have stopped here, and that this is true, will be demonstrated in this section. Those familiar with the Kepler Problem might recall that, although Kepler could not himself achieve any explicit solution to the problem of the relationship between the mean anomaly to the eccentric anomaly, he was not entirely without means for arriving at practical results. In fact, we know from his publication of the Rudolphine Tables that, even with this paradoxical knowledge of the relationship between time and the motion of a planet, he was still fully capable of determining the positions of all the planets to a degree of accuracy never before achieved.

From the time of Kepler to Gauss’ day, no fundamental advances in solving the Kepler Problem had been made. The only methods popularly considered to be advances over the ones Kepler used, were the very same that Olbers had brought up to Gauss, those of Lambert and Lagrange, which reduced the Kepler Problem to an infinite series. Ironically, however, it was Gauss himself who, in his Theoria Motus, §11, made the point that the infinite series solutions were actually more impractical and less convenient to calculate than those that

\(^{18}\)Here $x = \sin^2 \frac{1}{2}g$ and $X = \frac{2g - \sin 2g}{\sin^3 g}$.
Kepler had suggested.\(^\text{19}\)

Thus, let us take up an application of such a method in solving the equation we have at hand. Gauss broadly describes the method for solving the Kepler Problem as follows:

\[
\text{[The Equation], } E = M + e \sin E, \text{ which is to be referred to the class of transcendental equations, and admits no solution by means of direct and complete methods, must be solved by trial, beginning with an approximate value of } E, \text{ which is corrected by suitable methods repeated often enough to satisfy the preceding equation, that is, either with all the accuracy the tables of sines admit, or at least with sufficient accuracy for the end in view. If now, these corrections are introduced, not at random, but according to a safe and established rule, there is scarcely any essential distinction between such an indirect method and the solution by series, except that in the former the first value of the unknown quantity is in a measure arbitrary, which is rather to be considered an advantage since a value suitably chosen allows the corrections to be made with remarkable rapidity.}
\]

*Theoria Motus*, §11

Let us now apply this to our example above. To restate our data, we had taken the radii for the first and third observations to be \( r = 2.7337947 \), \( r' = 2.7685475 \), and the angle between them as \( \theta = 8^\circ 49'24''09 \) or, expressed in parts of the radius, \( \theta = 0.1539967 \). We had calculated our time elapsed (from the observation times) to be 40.8949120 days.

To begin, we should calculate our value for \( m \), since this is what we will be adjusting toward. If all things go well (remember to convert the time elapsed to years) our value should be \( m = 0.05474897 \).

Next, we need an approximate value of \( g \). Where can we find one? How about from the difference of the true anomalies \( f \)? The attentive reader might already know that this is an especially good guess in our circumstances.\(^\text{20}\)

Denoting our first approximation by \( g_0 \), we will take \( g_0 = 4^\circ 24'42''.045 \) or, expressed in parts of the radius, \( g_0 = 0.07699835 \), will yield us a value for \( m \) equal to 0.05478176, which will denote by \( m_0 \). This is already very close, having a difference with the true value of only \( \Delta m_0 = m_0 - m = 0.00003279 \).

Now we need to determine how much we should adjust \( g_0 \) in order achieve a change in \( m \) equivalent to \( \Delta m_0 \). How can this be done? Can we know how much a change in \( g \) will effect \( m \)? Although we may be able to determine this over any finite interval, we can use as an approximation, the rate of change at

\(^{19}\)Gauss effectively restored Kepler’s method for solving the Kepler Problem, with one slight modification in order that logarithmic tables might more readily be employed.

\(^{20}\)Although choosing \( f \) as a first guess for \( g \) will, under most circumstances, be a good starting point, we leave it to the reader to hypothesize in what situations it would be best to adjust this value to a greater or lesser one.
the position \( g_0 \), that is, \( \frac{dm}{dg} \). By differentiating the function\(^{21}\) and evaluating it for \( g_0 \), the rate of change will be 0.35665044.

Thence, since \( \Delta m_0 \) is the amount of change in \( m \) required, and \( \frac{dm}{dg} \) is the amount of change that occurs in \( m \) at \( g_0 \), by dividing the former by the latter we should obtain an approximate value for the change needed in \( g_0 \), namely, \( \Delta g_0 = 0.00009195 \). With this, we will now have a new value \( g_1 = g_0 - \Delta g_0 = 0.07690640 \).

Repeating the process with this new value, we obtain \( m_1 = 0.05474898 \), which differs from \( m \) by only a single digit in the eighth decimal place!

Since we have the advantage of modern computing systems, rather than calculating by hand, we can easily follow through another adjustment to obtain

\[
g = 0.07690638, \text{ or } 4^\circ24'23''.079
\]

which gives us \( m \) exactly.

### 6.1 Determination

Where to now? Well, we still have a planet to catch! How does this value of \( g \) help us attain our determined goal? If we think back to all that hard work we accomplished, we will recall that in the setup we had arrived at many elegant expressions relating the true anomalies, the eccentric anomalies, and the radii. Also involved in those expressions were the semi-axes major and minor. With these as our tools, the orbit is easily captured.

For instance, taking the relation

\[
b \sin g = \sqrt{rr'} \sin f,
\]

we now know that the semi-axis minor must be 2.75439986.

From the relation

\[
a = \frac{r + r' - 2\sqrt{rr'} \cos f \cos g}{2 \sin^2 g}
\]

we find the semi-axis major to be 2.76369809.

Since these two have the relation \( b = a \cos \phi \), we can easily obtain the values

\[
\cos \phi = \frac{b}{a} = 0.99663559
\]

\[
e = \sqrt{1 - \cos^2 \phi} = 0.081960419
\]

\[
p = b \cos^2 \phi = 2.74513292
\]

\(^{21}\)We will not do this here, but if the reader carries it out, they should obtain the equation

\[
\frac{dm}{dg} = \frac{1}{4} \sin g (l + \sin^2 \frac{1}{2} g) - \frac{1}{2} \left( l + \sin^2 \frac{1}{2} g \right)^2 \left( \frac{2g - \sin 2g}{\sin^2 g} \right) + (l + \sin^2 \frac{1}{2} g) \left( \frac{2 \sin g - 6g \cdot \cos g + 3 \sin 2g \cos g - 2 \cos 2g \sin g}{\sin^4 g} \right).
\]
The astute reader might notice that these values differ slightly from those we calculated at the beginning, when we assumed the semi-parameter = 2.7451305. Perhaps some error crept in somewhere?

If the reader apply his skill to check what these values of the elements yield for the calculated time elapsed between the observations, they will find the surprising result: 40.8949112 days. That is, no error whatsoever!\footnote{Perhaps ’no error’ is an overstatement: the actual error amounts to 0.0000019 seconds, or about one second every seventy-thousand years!}

What need had Gauss to improve upon this method? Could he not have stopped here, ending this section of the *Theoria Motus*:

> It is enough for me to believe that I could not solve this *a priori*, owing to the heterogeneity of the arc and the sine. Anyone who shows me my error and points the way will be for me the great *Kepler*.

### 7 If Not Geometric, Then What?

[Part II coming soon!]