On the Solution
of
Problems of Geometry
Using the Simplest Curves
and Dealing In Particular with each Class of Problems
A Dissertation in Three Parts[1]

Pierre de Fermat

First Part

It could be a paradox to say that, even in Geometry, Descartes was only a man; but, in order to determine whether this is true, the most subtle Cartesians seek out whether there is an imperfection in the distribution their master made of curved lines into certain classes and degrees, and whether it were not better to adopt a more satisfying classification, one more suitable to the true laws of Geometric Analysis. We believe that we will be able to draw more from this question without in any way diminishing the glory of such an illustrious man, for it is in the interest of Descartes and all the Cartesians, that the Truth (to which they rightly cleave, being Its most outspoken partisans, although sometimes Truth disagrees with their opinions) will become manifest for all, or, if this expression is too general, at least for geometers and analysts.

The distribution of problems of Geometry into determinate classes seemed necessary, not only to the ancients, but also to modern analysts. Let us first pose the equations

$$a + d = b \qquad \text{or} \qquad a^2 + ba = z''.$$

In the terms of the first, the unknown does not exceed the root or the side, while in the second we find the second power, or the square of the unknown side. Both taken together constitute the first, and simplest, species of problems. These are the problems which geometers are accustomed to call *plane* problems.

The second species of problem is that wherein the unknown quantity is raised to the third or fourth power, i.e. to the cube or biquadrate. The reason that two consecutive powers, differing in their degree, nonetheless constitute one and the same species of problem, is that quadratic equations are easily reduced to simple or linear equations, by a process known to both the ancients and the moderns, and which can be easily performed with straight-edge and compass. Similarly, biquadratic equations of the fourth degree can be reduced to cubic equations of the

---

[1] Originally written in Latin under the title *De solutione problematum geometricorum per curvas simplicissimas et unicuique problematum generi proprie convenientes, dissertatio tripartita*. This translation was made by Jason Ross from the French translation found in the *Œuvres*, Volume III.

third degree, by the method given by Viète and Descartes. Indeed, it is the object of Viète's subtle *paraplérose climatique* that can be seen in is treatise *De Emendatione Æquationum,* Chap. VI, and the artifice that Descartes makes use of in similar cases is exactly the same, although he expresses the terms differently.

In the same way, an analyst following Viète or Descartes could, although with more difficulty, reduce a bicubic equation to a quadratocubic equation, which is to say reducing an equation of the sixth degree to one of the fifth degree. Now from what has been seen in the aforesaid cases in which there was only one unknown, namely that equations of even degree can be reduced to equations of immediately lesser, odd degree, Descartes affirmed with confidence (page 323 of the *Géometrie* which he published in French) that it was absolutely the same for equations containing two unknowns: Such are all the equations constituting curves. Now, in these equations, not only does the reduction in question not succeed (as Descartes affirmed it would), but analysts have even considered it absolutely impossible. Let us consider, for example, the equation of the biquadratic parabola

$$a^4 = z'''e.$$

By what means may we reduce this fourth-degree equation to one of the third degree? What *paraplérose climatique* may we imagine?

Like Viète, I will designate unknown quantities by vowels, since I do not see why Descartes changed something without importance and which is purely a convention.[2]

This discussion or remark is neither made out of laziness, nor is it useless, since I am proving the general method by which all problems may be reduced to a curves of a certain degree.

If a problem were proposed wherein the unknown quantity were raised to the third or fourth power, we would be able to solve it using conic sections which are of the second degree. If the equation is raised to the fifth or sixth power, we can give the solution by means of curves of the third degree. If the equation rises to the seventh or eighth power, we could give the solution by using curves of the fourth degree, and so on infinitely by an identical process. It is clear from this that the question referred to above is not a question of words, but of the thing itself.

For example, let it be that

$$a^6 + b^{\mathrm{V}}a = z^{\mathrm{VI}}, \qquad \text{of, if you like,} \qquad a^5 + b^{\mathrm{IV}}a = z^{\mathrm{V}};$$

in both cases, the problem will be resolved by cubic curves of the third degree, just as Descartes has done with the rest.

But if it be posed that

$$a^8 + b^{\mathrm{VII}}a = z^{\mathrm{VIII}}, \qquad \text{or} \qquad a^7 + b^{\mathrm{VI}}a = z^{\mathrm{VII}},$$

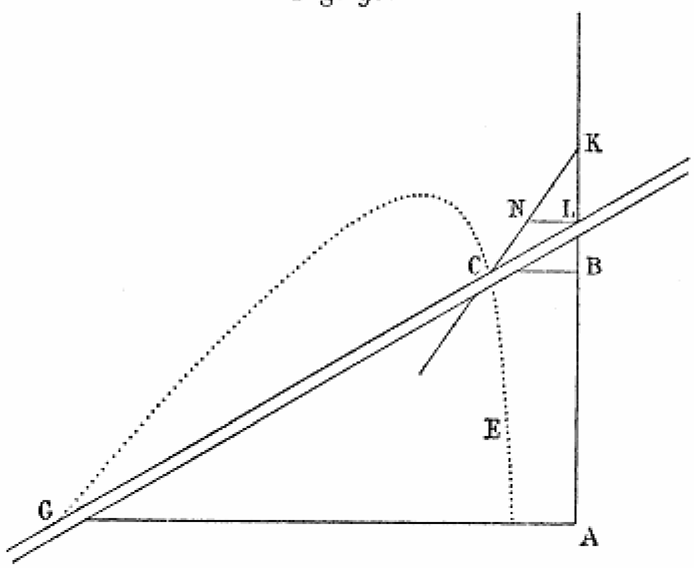
we could solve the problem by using biquadratic curves of the fourth degree, which Descartes neither accomplished, nor deemed possible, because he believed that in this case it was necessary to make use of curves of the fifth or sixth degree. Now it is a fault in true Geometry to solve a given problem by means of excessively complex curves of too high a degree, while passing by

[2] Instead of vowels, Descartes designated unknowns by x, y, z, etc. (All footnotes are the translator's.)

simpler and more suitable curves. Before the moderns, Pappus had already remarked that it is a real sin against the rules of Geometry to solve a problem by a species of curve which does not suit the problem. To avoid this error, we must correct Descartes and bring each problem to its particular and natural rank.

On page 322, Descartes again distinctly affirms that curves born from the intersection of a straight-edge and another line or curve, are always of a higher degree or species than the line or curve of the figure (page 321) from which they derive (*fig.* 90).

Fig. 90.



But let us imagine, for example, instead of the line CNK of said figure on page 321, a cubic parabola having K as its vertex and unknown KLBA as its axis; if we realize the construction in the spirit of Descartes, it is clear that the equation of this cubic parabola will be

$$a^3 = b^2e.$$

We will soon see that the curve EC arising from this assumption is only biquadratic; therefore the biquadratic has a degree higher by one than the cubic curve, according to the rule enunciated by Descartes himself, even though he expressly declares the contrary (on page 323), namely that the biquadratic and cubic curves are of the same degree or species.

Regarding our method which reduces all problems up to infinity, to wit: problems with equations of the third and fourth power reduced to curves of second degree; those of fifth and sixth power, to the third degree; those of the seventh and eighth power, to the fourth degree, and so on indefinitely, we will not hesitate to communicate it to all those who consider it wrong to conceal (to truth's disservice) any error whatsoever – even if it be an error of Descartes.

Let us not be hindered by the fact that problems of the second power, which are of the same species as those of the first degree (being called, like them, *plane* problems), require a circle, a curve of the second degree. This objection will be met with a special response when we give our general method for solving absolutely all problems by the curves which are suitable to them.

Second Part of the Dissertation

To satisfy the commitment that I made publicly, I give my general method for the solution of all problems by use of curves suitable to each.

I have already said, in the first Part of this Dissertation, that the problems of two immediately consecutive degrees, such as 3 and 4, 5 and 6, 7 and 8, 9 and 10, etc., each demand only one degree of curve. Thus those of the third and fourth power are solved by curves of the second degree, those of the fifth or sixth by curves of the third degree, and so on forever.

Here is the way it works. The given arbitrary equation containing only one unknown will, first of all, be brought to the higher degree, that is, to the even degree. Then we will remove the term containing the unknown of the first degree. This done, there will remain an equation between the known quantity (the given term) on one side, and on the other there will be an unknown in each term into which the square of the unknown root will enter. We will equate this unknown term to a square whose root is formed such that by equating the aforesaid square with the unknown term, we may eliminate as many of the highest degrees of the unknown root. Moreover, it is necessary to take care that all terms of the root of the square we are making be affected by the unknown root (quantity). Care must also be taken that the last of these terms also be affected by a second unknown. It would then happen that by a simple division of one side, and by the extraction of a square root of the other, two equations of curves suitable to the given problem would be created, and their intersection will solve the problem by the method that we have long applied to the solution of problems by loci.

For example, let: $a^6 + ba^5 + z^{II}a^4 + d^{III}a^3 + m^{IV}a^2 = n^{VI}$.

All problems which rise to the fifth or sixth power can be brought to this form. For it suffices for this either to bring the fifth to the sixth power, or to remove the a term from an equation of sixth power. These are things sufficiently taught by Viète and Descartes.

We will form the square of the root: $a^3 + bae$, and we will equate it with the left hand side of the given equation. We will then have

$$a^6 + 2ba^5e + b^2a^2e^2 = a^6 + ba^5 + z^{II}a^4 + d^{III}a^3 + m^{IV}a^2.$$

Removing a^6 from both sides and dividing by a^2, which can always be performed if the indicated precaution was taken in using this method, there remains the equation

$$ba^3 + z^{II}a^2 + d^{III}a + m^{IV} = 2ba^2e + b^2e^2,$$

which, as can be seen, gives a curve of the third degree.

But, to have the second equation and to easily arrive at the solution of the problem, it is necessary to equate the other side of the equation n^{VI} to the square of $a^3 + bae$.

Therefore, by extracting the root and calling, for example, n^{III} the square root of n^{VI}, which is easily done, we will have

$n^{III} = a^3 + bae$, which is the root of the square made equal to the left hand side of the original equation.

We therefore have a second equation which also gives a curve of the third degree. Who does not now see that the intersection of the two found curves will give the value of *a*, which is the solution of the proposed problem?

If the problem rises to the seventh or eighth power, we will first put it in the form of an equation of eighth degree, and then remove from it the term which has only the root. After this permitted reduction, which is conventional for the method, let us have:

$$a^8 + ba^7 + d^{II}a^6 + n^{III}a^5 + m^{IV}a^4 + g^{V}a^3 + r^{VI}a^2 = z^{VIII}.$$

We will form the square to which we will equate the two sides of this equation, by using the root

$$a^4 + \tfrac{1}{2}ba^3 + d^{II}ae.$$

I have formed the second term of this root of the square in such a way that the two highest powers of the unknown *a* will be eliminated in the equation, which is quite easy. By equating the square of this root to the left hand side of the proposed equation, and removing the common terms and dividing by a^2, we will have on one side an equation representing a curve of the fourth degree. Then we will take the square root of the right hand side of the originally proposed equation; let p^{IV} be this root of z^{VIII}, which we will equate to $a^4 + \tfrac{1}{2}ba^3 + d^{II}ae$. This equation will give another curve of the fourth degree and the intersection of these two curves will give the value of *a*, the solution of the proposed problem.

It is necessary to remark for the remainder, that in problems which rise to the power of 9 and 10, it is necessary to form the root of the square in such a way that it contains at least four terms, in order to eliminate the three highest degrees of the unknown.

For problems of the 11th and 12th powers, the root of the square must have at least five terms, and must be created in order to eliminate the four largest degrees of the unknown.

The process for thus forming the root is still quite simple; the analysts will find by trial that it is absolutely sufficient for the division or application (to employ geometric terms in a purely geometric subject); the signs + and – will cause no difficulty in practicing this method.

Since, moreover, problems which rise to the second degree are reduced to the first degree by extraction of the square root, this method gives their solution as known by means of lines of the first degree, i.e. straight lines; thus we see the ineffective objection of which we have spoken in the first Part of this Dissertation vanish, if the extraction of the square root be assumed to be immediately known for all types of problems.

Thus, we have the most exact and simplest possible solution and construction for the problems of Geometry by using loci [[par les lieux]], emerging from the cases of curves of different species and suited to these problems. [[]] For the remainder, the analyst will be free to change these curves, while remaining always in the species natural to the problem: by resolving those of the 8th or 7th degrees by curves of the 4th; those of the 10th and 9th by curves of the 5th; those of the 12th and 11th by curves of the 6th; and so on indefinitely by a uniform method. On the contrary, according to Descartes, problems of the 8th and 7th degrees must be solved by curves of the 5th and 6th, problems of the 10th and 9th by curves of the 7th or 8th, problems of the 12th or 11th by curves of the 9th or 10th, and so on indefinitely. The Cartesians can see how far from simplicity and geometric truth this is, or, if it be their desire, they may try to contradict us.

Because we are seeking the truth, if it is hidden somewhere in the writings of this great man, we will have the greatest joy in coming to know it and embracing it; because, to employ a formulation which is not my own, I can say that I have such a great admiration for this extraordinary genius, that I have more respect for Descartes when he errs than I have for many others when they are correct.

THIRD PART OF THE DISSERTATION

This may suffice for the general theory; for the problems that Descartes considers to be solvable by means of curves of an unnecessarily high degree, we have fortunately lowered them by a general method to curves of a degree half smaller. But this must be understood to mean that a smaller degree is required for absolutely all questions, because an infinity of special cases lend themselves to a still greater reduction. I would therefore like to go further still, and bring Cartesian analysis, not only to terms of half the degree, but to degrees 4 times less, 6 times less, 100 times less, and indefinitely less elevated for certain cases. Thus we will be better able to recognize the error of Descartes, and it will find its immediate correction by analysis. For convenience, in what follows, I shall designate the powers of higher degree by the numbers which their exponents bear.[3]

Let it be proposed *to find six mean proportionals between two given magnitudes.* Let b and d be the two givens, and let a the first mean to find, from which we have the equation $a^7 = b^6d$. According to Descartes, this equation can only be resolved by curves of the 5th or 6th degree. In the Second Part of this Dissertation, this equation, like all others of the same nature, is solved generally by curves of the 4th degree. But nothing prevents our solving it by curves of the 3rd degree. As a matter of fact, let us equate each of the sides of the equation to the term a^4e^2d. If, in the equation containing a^7, we divide both sides by a^4, there results $e^2d = a^3$, which, as we see, gives a curve of 3rd degree. On the other side, $a^4e^2d = b^6d$; dividing by d and taking the square root, we arrive at $a^2e = b^3$, which also gives a curve of the 3rd degree. The intersection of these two curves will give the value of a, that is, the solution of the proposed problem by means of 3rd degree curves.

Now let it be proposed *to find twelve proportional means between two given magnitudes.* The equation will be $a^{13} = b^{12}d$. Descartes believed that it could be solved only by curves of the 11th or 12th degree. In the Second Part of this Dissertation, I have taught that all equations of this degree can be solved by curves of the 7th degree. But a more attentive examination immediately gives an elegant solution through the use of curves of the 5th. It can even be obtained by curves of the 4th, as will be seen in the following.

[3] That is "6th degree," rather than "bicubic."

First, let us equate both sides to the term a^8e^4d. By taking the first equation with a^{13} and dividing both sides by a^8, we arrive at $a^5 = e^4d$, a curve of the fifth degree. In the second equation with $b^{12}d$, we divide by d and take the fourth (biquadratic) root, giving us $a^2e= b^3$, a curve of the third degree. The proposed problem is thus solved by two curves, one of the 5th degree, and the other of the 3rd degree.

But we may solve this question still more easily, namely with curves of the 4th degree. If, in fact, we equate the two sides to a^9e^3d, we will have on one side, after dividing by a^9, $a^4 = e^3d$, the equation of a curve of 4th degree; by dividing the other side by d and taking the third (cubic) root, we have $a^3e = b^4$, which also gives a curve of the fourth degree. Thus we have a simple construction by two curves of the 4th degree.

Following these examples, one may not doubt that the *creation of 30 proportional means* can be obtained by curves of the 7th degree or even the 6th degree. Let us equate the two sides of the equation $a^{31} = b^{30}d$ to the common term $a^{24}e^6d$; the problem will be brought to curves of the 7th degree. By using $a^{25}e^5d$, it will be brought to curves of the 6th.

Similarly, the *creation of 72 proportional means* can be made by curves of the 9th degree, and it is clear, following what has preceded, that it is possible to assign a ratio greater than any given ratio between the degree of the problem and the degree of the curves which solve it. When the Cartesians will have seen this, I do not doubt that they will acknowledge the necessity of our remark and of our correction.

It is necessary to observe that it is often fitting to change the form of the equation so that the degree be susceptible of a convenient division into aliquot parts. It will be unnecessary to repeat this remark.

For example, let the *creation of 10 means* be proposed. This gives the equation $a^{11} = b^{10}d$. We will multiply both sides by a given line, such as z, giving the equation $a^{11}z = b^{10}dz$, and we have thus arrived at the number 12, which easily permits of a reduction to its aliquot parts. In equating each side of the equation to a^8a^4, we will have on one side $a^3z = e^4$, a curve of the fourth degree. On the other side, by taking the fourth root, which let be n''' for the term $b^{10}dz$, we have $a^2e = n'''$, a curve of the third degree. Thus we can find ten means by two curves, one of the 4th and the other of the 3rd degree, a result which we reached easily by a small change in the original equation.

I will not linger on the other abbreviations (of which there are an infinite number) which the art will furnish to analysts themselves. All the same, I add that what I have just said can be applied not only when the unknown is found without any other term affected by it to a lower degree, but also if there are terms of neighboring higher degree, as in the equation $a^{13} + na^{12} + ma^{11} + ra^{10} = b^{12}d$.

The solution of this equation, made by taking the same common term as above, a^9e^3a, will be just as easy as finding 12 means between two given extremes. The same technique is also employed for equations of other higher degrees.

However, it must be remarked that, in equations where only one unknown term is found on one of its sides, it must be that the exponent of unique power of the unknown be a prime number, in order that one may designate the degree of the problem by that power. If, indeed, the exponent is composite, the problem is immediately brought to the degree of its divisors.

For example, if the question were posed of finding 8 proportional means between two given magnitudes, the equation would be $a^9 = b^8 d$. In this case, the number 9 being composite and having 3 as its factor twice, the problem must be regarded as one of 3rd degree, and indeed it is. In fact, if we find two proportional means between two givens, and then intercalate two new proportional means between the first and second terms of the sequence we have formed, and then do the same between the second and third, and the third and fourth, then we will have 8 means between the two lines originally proposed.

If now 14 means between two givens are required, the equation $a^{15} = b^{14} d$ shows that the problem is reduced to two others, one of the 3rd degree and one of the 5th degree.

Thus it is seen that the exponent of the unique power must be a prime number in order to express and truly represent the degree of the problem.

Moreover, since I consider as a given that *numbers obtained by adding unity to successive squares are always prime beyond 2*, a theorem whose truth I have long ago announced to analysts, I would like to say that the numbers

$$3,\ 5,\ 17,\ 257,\ 65537,\ \ldots,\ \text{to infinity}$$

are prime. There is no difficulty in finding a process allowing the *construction of a problem whose degree has a ratio greater than any given ratio to the degree of the curves which serve to solve it.*

For example, let the problem be posed of finding 256 proportional means between two extremes. The equation would be $a^{257} = b^{256} d$, which, by equating both sides to $a^{240} e^{16} d$, shows that the question can be solved by curves of the 17th degree.

If we sought 65536 means, the problem would be solved by curves of the 257th degree, and so on indefinitely, reducing the degree from the greater number to that of the immediately preceding number. And who does not see that between two consecutive numbers, the ratio increases indefinitely?

Will the Cartesians still try to conceal the error of Descartes? As for me, I refrain from making any forecast; I await with interest to see, but say nothing further regarding, the future fate of this subject.