METHOD
FOR THE
STUDY OF MAXIMA AND MINIMA[1]
Pierre de Fermat

The whole theory of the study of maxima and minima assumes the position of two unknowns and this sole rule:

Let a be an arbitrarily chosen unknown of the question (whether it has one, two, or three dimensions, as follows from the statement). We will express the maximum or minimum quantity in terms of a, by means of terms of any degree. We will then substitute $a+e$ for the primitive unknown a, and express the maximum or minimum quantity in terms containing a and e to any degree. We will *ad-equate*, to speak like Diophantus, the two expressions of the maximum and minimum quantity, and we will remove from them the terms common to both sides. Having done this, it will be found that on both sides, all the terms will involve e or a power of e. We will divide all the terms by e, or by a higher power of e, such that on at least one of the sides, e will disappear entirely. We will then eliminate all the terms where e (or one of its powers) still exists, and we will consider the others equal, or if nothing remains on one of the sides, we will equate the added terms with the subtracted terms, which comes to be the same. Solving this last equation will give the value of a, which will lead to the maximum or the minimum, in the original expression.

Let us take an example:

Divide the line AC (*fig.* 91) *at* E, *such that* AE×EC *be a maximum.*

Fig. 91.

Let us take AC = b; let a be one of the segments, and let the other be $b - a$, and the product whose maximum we have to find is: $ba-a^2$. Now let $a+e$ be the first segment of b, the second $b-a-e$, and the product of the two segments will be: $ba-a^2+be-2ae-e^2$.

It must be *co-equal to the preceding:* $ba-a^2$;
Removing the common terms: $be \sim 2ae + e^2$;
Dividing all the terms: $b \sim 2a + e$;
Remove e: $b = 2a$.

To solve the problem, therefore, the half of b must be taken.

It is impossible to give a more general method.

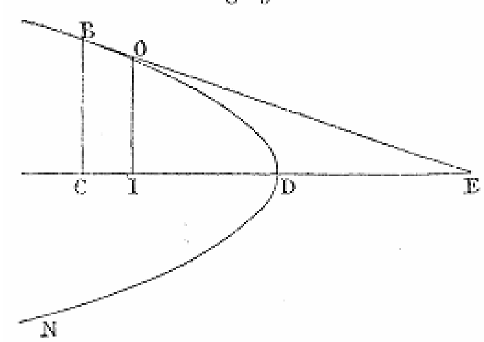
---

[1] Originally written in Latin, the *Methodus ad Disquirendam Maximam et Minimam*, was sent, via Mersenne, to Descartes, who received it around the 10th of January, 1638. This English translation was made by Jason Ross from the French translation in the Œuvres de Fermat, vol. 3, pp. 121-156.

The invention of tangents at given points on arbitrary curves is reducible to the preceding method.

Let there be, for example, the parabola BDN (*fig*. 92), with vertex D and axis DC;

Fig. 92.



let B be a given point on it, through which the straight line BE is drawn tangent to the parabola and touches the axis at E.

If we take an arbitrary point O on the line BE, from which the ordinate OI is drawn, just as the ordinate BC from point B, we will have: $CD/DI > BC^2/OI^2$, since the point O is outside the parabola. But $BC^2/OI^2 = CE^2/IE^2$, because of the similarity of the triangles. Therefore $CD/DI > CE^2/IE^2$.

Now point B is given, and therefore the ordinate BC, and therefore point C and line CD. Therefore, let CD = *d* be given. Let us set CE = *a* and CI = *e*; we will have $d/(d-e) > a^2/(a^2+e^2-2ae)$.

Let us make the product of the means and of the extremes:[2]

$$da^2 + de^2 - 2dae > da^2 - a^2e.$$

Co-equaling then, following the preceding method, by removing the common terms:

$$de^2 - 2dae \sim -a^2e,$$

or, which is the same thing:

$$de^2 + \underline{a}^2e \sim 2dae.$$

Divide all the terms by *e*:

$$de + a^2 \sim 2da.$$

Remove *de*: there remains $a^2 = 2da$, therefore: $a = 2d$.

Thus we prove that CE is double CD, which agrees with the truth of the matter.

This method never fails, and may be extended to many beautiful questions: thanks to it, we have found the centers of gravity of figures bounded by straight and curved lines, as well as the centers of solids and a number of other things which we will be able to take up elsewhere, if we have the leisure to do so.

As for the quadrature of areas bounded by curved and straight lines, or when it comes to the ratio which the solids they create have with cones of the same base and height, we have already treated in detail with M. de Roberval.

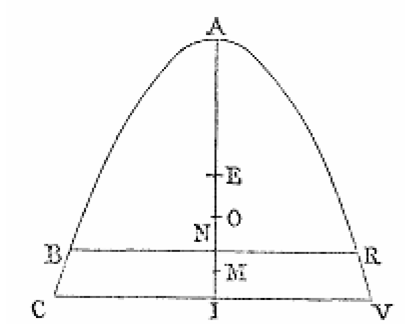
---

[2] “cross-multiplying” in today’s parlance

II
Center of Gravity of Parabolic Conoid[3]
Following the Same Method

Let CBAV (*fig.* 93) be a parabolic conoid, having IA as its axis and having a circle of diameter CIV as its base. Let us find its center of gravity by the very same method, which served us for finding the maxima, minima, and tangents of curved lines, and let us therefore prove here, by new examples and by a new and brilliant employment of this method, the error of those who believe that it is defective.

Fig. 93.



To arrive at the analysis, let us say that IA = b. Let O be the center of gravity; let us call a the unknown length AO; cut the axis IA by an arbitrary plane BN, and let IN = e, from which NA = $b - e$.

It is clear that in this and similar figures (parabola or parabolic), the centers of gravity, in the segments removed by parallels to the base, divide the axes in a constant ratio (it is clear, in fact, that the demonstration of Archimedes for the parabola may be extended, by an identical reasoning, to all parabolas and parabolic conoids). Thus the center of gravity of the segment, of which NA is the axis and BN the radius of the base, will divide AN in a point, such as E, such that NA/AE = IA/AO, or, in notation, $b/a=(b-e)/\mathrm{AE}$.

The portion of the axis will thus be AE = $(ba-ae)/b$, and the interval between the two centers of gravity, OE = ae/b.

Let M be the center of gravity of the remaining portion CBRV; it must necessarily fall between the points N and I, inside the figure, according to postulate 9 of Archimedes in his *De æquiponderantibus*, since CBRV is an entirely concave figure in regards to its interior.

But partCBRV/partBAR = EO/OM, since O is the center of gravity of the total figure CAV and E and M the centers of gravity of the parts.

Now in the conoid of Archimedes, partCAV/partBAR = $\mathrm{IA}^2/\mathrm{NA}^2 = b^2/(b^2+e^2-2be)$; therefore, *dividendo*: partCBRV/partBAR = $(2be - e^2)/(b^2+e^2-2be)$. But we have proved that

---

[3] This section appears to be that which Fermat addressed to Mersenne for Roberval, with his letter of April 20, 1638.

partCBRV/partBAR = OE/OM. Therefore, in our notation, $(2be\text{-}e^2)/(b^2+e^2-2be)$ = $(OE(=ae/b))/OM$; whence $OM = (b^2ae + ae^3 - 2bae^2)/(2b^2e - be^2)$.

Following what has been established, the point M is between points N and I; therefore OM < OI; then in notation, $OI = b - a$. The question thus comes back to our method, and we may suppose

$$b - a \sim (b^2ae+ae^3-2bae^2)/(2b^2e\text{-}be^2).$$

Multiplying both sides by the denominator, and dividing by e:

$$2b^3 - 2b^2a - \underline{b}^2e + bae \sim b^2a + ae^2 - 2bae.$$

Since there are no common terms, let us remove all those which contain e and let us consider the others as equal:

$$2b^3 - 2b^2a = b^2a\text{, whence } 3a = 2b.$$

Consequently, IA/AO = 3/2, and AO/OI = 2/1. Q.E.D.

The same method may be applied to the centers of gravity of all parabolas to infinity, just as it may be applied to parabolic conoids. I do not have the time to indicate, for example, how one could find the centers of gravity in *our parabolic conoid of revolution around the ordinate*; it would suffice to say that in this conoid, the center of gravity would divide the axis into two segments which are in the ratio 11/5.

III
On the Same Method

By means of my method, I would like to *divide a given line* AC (*fig.* 94) *at point* B, *such that* $AB^2 \times BC$ *be the maximum* of all solids which could be formed in the same fashion by dividing the line AC.

Fig. 94.

Let us suppose, in algebraic notation, that AC = b, the unknown AB = a; we will have BC = $b-a$, and the solid $a^2b - a^3$ must satisfy the proposed condition.

Now taking $a+e$ in place of a, we will have for the solid

$$(a+e)^2(b-e-a) = ba^2 + be^2 + 2bae - a^3 - 3ae^2 - 3a^2e - e^3.$$

I compare this to the first solid: $a^2b - a^3$, as if they were equal, when in fact they are not. It is this comparison that I call *adequality*, to speak as Diophantus, for one can thus translate the Greek word παρισότης that he uses.

Then, I subtract the common terms from both sides, *viz.*, $ba^2 - a^3$. This done, one side of the equation has nothing, while the other is $be^2 + 2bae - 3ae^2 - 3a^2e - e^3$. Therefore, we must compare the positive terms with the negative ones; we thus have a second *adequality* between $be^2 + 2bae$ on one side, and $3ae^2 + 3a^2e + e^3$ on the other. Dividing all the terms by e, the *adequality* will hold between $be + 2ba$ and $3ae + 3a^2 + e^2$. After this division, if all the terms may again be divided by e, the division must be repeated, until there be a term that can no longer be divided by e, or, to employ the terminology of Viète, a term which is no longer affected by *e*. But, in the proposed example, we find that the division cannot be repeated; so, we have to stop there.

Now, I remove all the terms affected by *e*; on one side there remains $2ba$, while the other has $3a^2$, terms between which it is necessary to establish not a feigned comparison or an *adequality*, but rather a true equation. I divide both sides by a; giving me $2b = 3a$, or $b/a = 3/2$.

Let us return to our question, and divide AC at B such that AC/AB = 3/2. I say that the solid $AB^2 \times BC$ is the maximum of all those which can be formed by dividing line AC.

To establish the certitude of this method, I will take an example from the book of Apollonius, *On the Determined Section*, which according to the account of Pappus (at the beginning of Book VII), holds difficult limitations and notably that which follows, which I consider to be the most difficult. Pappus (Book VII) assumes it to be found, and, without demonstrating it to be true, considers it such, and derives other consequences from it. In this location, Pappus calls a minimum ratio μοναχὸν καὶ ἐλάχιστον (singular and minimum), because, if one poses a question regarding given magnitudes which is satisfied in general by two points, then for the maximum or minimum values there would only be one point. It is for this reason that Pappus calls the smallest possible ratio for the question *minimum* and *singular* (that is, unique). On this point, Commandino doubts the significance of Pappus's term μοναχὸς, but this is because he ignores the truth that I have just stated.

Here is the proposition. – *Let a line* OMID (*fig.* 95) *be given, and on it four given points* O, M, I, *and* D. *Divide the segment* MI *at point* N, *such that* (ON×ND)/(MN×NI)[4] *be a smaller ratio than that of any two other rectangles* (ON×ND)/(MN×NI).

Fig. 95.

Let us call the givens OM = b, DM = z, MI = g, and let the unknown MN = a. We will then have, in the form of notation:

$$ON \times ND = bz - ba + za - a^2, \quad MN \times NI = ga - a^2.$$

Therefore, it is necessary that the ratio $(bz - ba + za - a^2)/(ga - a^2)$ be the smallest of all those that can be obtained by any division of the line MI.

Now substitute $a + e$ for a, and we will have the ratio

$$(bz - ba - be + za + ze - a^2 - e^2 - 2ae) \,/\, (ga + ge - a^2 - e^2 - 2ae),$$

which must be compared by *adequality* to the first, which is to say that we will multiply on one side the first term by the fourth, and on the other, we will multiply the second by the third, and then compare the two products:

$$\underbrace{(bz - ba + za - a^2)}_{\text{First term}}\ \underbrace{(ga + ge - a^2 - e^2 - 2ae)}_{\text{last term}}$$

$$= bzga - gba^2 + gza^2 - ga^3 + bzge - bage$$
$$+ zage - a^2ge - bza^2 + ba^3 - za^3 + a^4$$

---

[4] The Œuvres publication incorrectly has MI instead of NI.

$$- bze^2 + bae^2 - zae^2 + a^2e^2 - 2bzae$$
$$+ 2ba^2e - 2za^2e + 2a^3e.$$

The other side,

$$\underset{\text{Second term}}{(ga - a^2)}\ \underset{\text{Third term}}{(bz - ba - be + za + ze - a^2 - e^2 - 2ae)}$$

$$= bzga - gba^2 - gbae + gza^2 + gzae$$
$$- ga^3 - gae^2 - 2ga^2e - bza^2 + ba^3$$
$$+ ba^2e - za^3 - za^2e + a^4 + a^2e^2 + 2a^3e.$$

I compare these two products by *adequality*; removing the common terms and dividing by *e*, we have:

$$bzg - a^2g - bze + bae - zae - 2bza + 2ba^2 - 2za^2$$
$$\sim -gae - 2ga^2 + ba^2 - za^2.$$

Removing all the terms in which *e* still exists, there remains:

$$bzg - a^2g - 2bza - 2za^2 + 2ba^2 = - 2ga^2 + ba^2 - za^2,$$

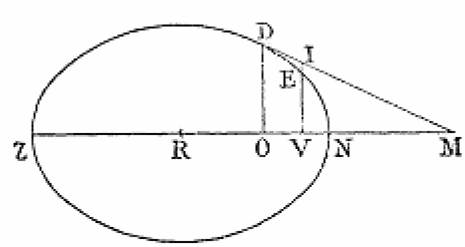
which becomes, by transposition,

$$- ba^2 + za^2 - ga^2 + 2bza = bzg.$$

By solving this equation, we will find the value of *a* or of MN, then the point N, and we will verify the proposition of Pappus, which teaches us that to find point N, we must make OM.MD / OI.ID = MN^2/NI^2; for the resolution of the equation will lead us to the same construction.

To also apply this same method to *tangents*, I can proceed as follows. For example, let there be an ellipse ZDN (*fig.* 96), with axis ZN and center R. Let us take on its circumference a point D, and draw the tangent DM to the ellipse from this point, and let us draw ordinate DO. In algebraic notation, let us suppose the given OZ = *b* and the given ON = *g*; let the unknown OM = *a*, understanding by OM the portion of the axis contained between point O and the point where the diameter reaches the tangent.

Fig. 96.



Since DM is tangent to the ellipse, if, through a point V taken at liberty between O and N, we draw IEV parallel to DO, then it is evident that the line IEV intersects the tangent DM and the ellipse, at points I and E. But, because DM is tangent to the ellipse, all of its points, except D, are outside the ellipse. Therefore, IV > EV and $DO^2/EV^2 > DO^2/IV^2$. But, following the property of the ellipse, $DO^2/EV^2 = ZO\cdot ON/ZV\cdot VN$, and $DO^2/IV^2 = OM^2/VM^2$. Therefore $ZO\cdot ON/ZV\cdot VN > OM^2/VM^2$.

Let the arbitrary OV = *e*. We have

$$ZO\cdot ON = bg, \qquad ZV\cdot VN = bg - be + ge - e^2,$$

$$OM^2 = a^2, \qquad VM^2 = a^2 + e^2 - 2ae.$$

Therefore $bg/(bg - be + ge - e^2) > a^2/(a^2 + e^2 - 2ae)$. If therefore we multiply the first term by the last and the second by the third, we will have

$$\underset{\text{(Product of the first term by the last)}}{bga^2 + bge^2 - 2bgae} > bga^2 - bae^2 + gea^2 - a^2e^2.$$

Following this method, it is therefore necessary to ad-equate these two products, remove the common terms, and divide the remainder by e; we then have

$$bge - 2bga \sim -ba^2 + ga^2 - a^2e.$$

Removing the terms which still contain e,

$$-2bga \sim -ba^2 + ga^2,$$

terms which must be made equal, following the method. Transposing as is necessary, we will have $ba - ga = 2bg$.

We see that this solution is the same as that of Apollonius, for, following the construction, to find the tangent, we must make $(b - g)/g = 2b/a$ or (ZO-ON)/ON=2ZO/OM, whereas according to Apollonius, we must make ZO/ON=ZM/MN. It is clear that these two constructions reach the same end.

I could add a number of other examples, both of the first and second cases of my method, but these are enough, and they prove sufficiently that it is general and never fails.

I neither add the demonstration of the rule, nor the numerous other applications which could confirm the high value of the rule, such as the discovery of centers of gravity and asymptotes, of which I have sent an example to the savant M. de Roberval.

IV
The Method of Maximum and Minimum

While studying the method of *syncrise* and *anastrophe* of Viète, and by carefully following its application to the study of the constitution of correlated equations, it came to my mind to derive from it a process to find the maximum and minimum, and thus to easily resolve all the difficulties surrounding limiting conditions, which have caused so much difficulty to ancient and modern geometers.

The maxima and minima are indeed unique and singular, as Pappus has said and as the ancients already knew, even though Commandino claimed ignorance of what Pappus meant by the term μοναχός (singular). It follows that on both sides of the limit point, one could find an ambiguous equation; that the two ambiguous equations are then correlative, equal, and alike.

For example, propose *dividing the line b such that the product of its segments is maximum*. The point satisfying this question is plainly the middle of the given line, and the maximum product is equal to $b^2/4$; no other division of this line will give a product equal to $b^2/4$.

But if one proposed dividing the same line b such that the product of its segments be equal to z^{II} (this area being assumed to be less than $b^2/4$), we will have two points satisfying the question, and they will be situated on either side of the point corresponding to the maximum product.

Indeed, let a be one of the segments of the line b. We will then have $ba - a^2 = z^{II}$, an ambiguous equation, since for straight line a the value can be either of the two roots. Let the correlative equation therefore be $be - e^2 = z^{II}$. Let us compare these two methods following the method of Viète:

$$ba - be = a^2 - e^2.$$

Dividing both sides by $a - e$, there results

$$b = a + e;$$

further, the lengths a and e will be unequal.

If we take an area larger than z^{II}, while still less than $b^2/4$, the difference between lines a and e will be less than earlier, the points of division approaching closer to the point constituting the maximum product. The more the product of the segments increases, the more, on the contrary, the difference between a and e will shrink, until it vanishes completely at the division corresponding to the maximum product. In this case, there is only one unique and singular solution: the two quantities a and e become equal.

Now the method of Viète, applied to the two correlative equations above, has led us to the equality $b = a + e$; therefore, if $e = a$ (which will constantly happen at the point constituting the maximum or minimum), we will have, in the proposed case, $b = 2a$. That is to say that if we take the center of the line b, the product of the two segments will be maximum.

Let us take another example: *Divide the line b such that the product of the square of one of the segments times the remaining segment will be the maximum.*

Let a be one of the segments. We must maximize $ba^2 - a^3$. The correlative, equal and similar equation is $be^2 - e^3$. Let us compare the two equations using the method of Viète:

$$ba^2 - be^2 = a^3 - e^3;$$

dividing both sides by $a - e$, there arrives

$$ba + be = a^2 + ae + e^2,$$

which gives the composition of the correlative equations.

To find the maximum, let us make $e = a$; there arrives

$$2ba = 3a^2 \qquad \text{or} \qquad 2b = 3a;$$

the problem is solved.

However, since, in practice, divisions by a binomial are generally complicated and very taxing, it is preferable, when comparing correlative equations, to bring to light the differences of the roots, so as to only require one simple division by this difference.

Let us seek the maximum of $b^2a - a^3$. Following the rules of the preceding method, we must take $b^2e - e^3$ as the correlative equation. But since e, just as well as a, is an unknown, nothing prevents our designating it as $a + e$; we will then have

$$b^2a + b^2e - a^3 - e^3 - 3a^2e - 3e^2a = b^2a - a^3.$$

If we eliminate the equal terms, it is clear that all the remaining terms will be affected by the unknown e: those involving only a are the same on both sides. Thus we have

$$b^2e = e^3 + 3a^2e + 3e^2a,$$

and, by dividing all the terms by e, we have

$$b^2 = e^2 + 3a^2 + 3ae,$$

which gives the form of the two correlative equations in this form.

To find the maximum, we must equate the roots of the two equations, in order to satisfy the rules of the first method, from which our new process derives its reasoning and means of operation.

Thus we must equate $a + e$ with a, whence $e = 0$. But, using the form that we have found for correlative equations,

$$b^2 = e^2 + 3a^2 + 3ae;$$

we must remove, from this equation, all the terms affected by e, by reducing them to 0; there remains $b^2 = 3a^2$, an equation which will give the sought-for maximum of the product.

To show more completely the generality of this double method, let us consider new types of correlative equations which Viète has not treated and which we will borrow from the Book on the *Determined Section* of Apollonius (in Pappus, Book VII, prop. 61), in which the question of limiting conditions is expressly recognized as difficult by Pappus.

Let there be line BDEF (*fig.* 97), *on which points* B, D, E, *and* F *are given. Find point* N *between points* D *and* E, *such that the ratio of the products* BN×NF *and* DN×NE *is made a minimum.*

Fig. 97.

Let us call DE = b, DF = z, BD = d, DN = a; it is required to minimize the ratio $(dz - da + za - a^2)/(ba - a^2)$.

The similar and equal correlative ratio $(dz - de + ze - e^2)/(be - e^2)$, according to our first method. Let us make an equation of the products of the middle and extreme terms, giving us

$$dzbe - dze^2 - dabe + dae^2 + zabe - zae^2 - a^2be + a^2e^2$$
$$= dzba - dza^2 - deba + dea^2 + zeba - zea^2 - e^2ba + e^2a^2.$$

Removing all the similar terms and making the proper transpositions:

$$dzba - dzbe + dea^2 - dae^2 - zea^2 + zae^2 + a^2be - e^2ba = dza^2 - dze^2.$$

Dividing both sides by $a - e$ (which will be quite easy, if correlated terms are taken together; thus $(dzba - dzbe)/(a - e) = dzb$, and $(dea^2 - dae^2)/(a - e) = dae$, etc.; it is easy to order the correlated terms to secure these divisions), we will have, after the division,

$$dzb + dae - zae + bae = dza + dze,$$

which equation provides the form of the two correlative equations.

Following the method, to move from this form to the minimum we must make $e = a$, whence

$$dzb + da^2 - za^2 + ba^2 = 2dza;$$

the resolution of this equation will give the value of a, *for which the proposed ratio will be minimum.*

The analyst will not be stopped by the fact that this equation has two roots, for the one that he must take will betray itself, even if one would not want to know it. Even with equations having more than two roots, a more-or-less wise analyst could always use one or the other of our methods.

But it is clear, following the example that we have just treated, that the first of these two methods will, in general, be less easy to use, because of its repeated divisions by a binomial. We must therefore have recourse to the second which, although simply derived from the first, as I have said, will procure for dexterous analysts a surprising facility and numerous shortcuts. What is more, it can be applied with a far superior ease and elegance to the study of tangents, centers of gravity, asymptotes, and other similar questions.

It is thus with the same confidence as before, that I still affirm today that the study of maxima and minima comes down to this unique and general rule, whose happy success will always be legitimate and not due to chance, as some have thought.

Let a be an unknown (see page 121, lines 6 to the end)… *its first expression.*

If there is still someone who considers the success of this method due to serendipitous chance, he is welcome to try to come across a similar one.

As for those who do not approve of it, I pose to them this problem:

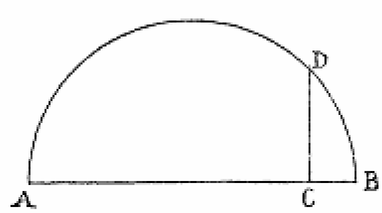
Given three points, find a fourth point such that the sum of its distances to the three given points be a minimum.

V
Appendix to the Method of Maximum and Minimum[5]

Radicals are often encountered in the course of working through problems. Therefore, the analyst must not hesitate to employ a third unknown, or, if necessary, to use a still greater number of them. In this manner, we shall in fact avoid elevations to powers which, in repeating themselves, usually complicate calculations. The technique of this method will be explained by the following examples.

Given a semicircle of diameter AB (*fig.* 98), *with perpendicular* DC *drawn upon its diameter, find the maximum of the sum* AC + CD.

Fig. 98.



[5] Written in 1644.

Let the diameter be taken as b, and let AC = a. We will thus have CD = $\sqrt{(ba - a^2)}$. The question becomes the maximization of the quantity $a + \sqrt{(ba - a^2)}$.

By applying the rules of the method, we will come to *ad-equating* expressions of excessive degree; let us then designate the maximum quantity by o;[6] for why should we abandon the custom established by Viète of using vowels to represent unknown quantities?

Thus, we will have $a + \sqrt{(ba - a^2)} = o$; therefore $o - a = \sqrt{(ba - a^2)}$, and by squaring, we will have:

$$o^2 + a^2 - 2oa = ba - a^2.$$

This done, we must make a transposition such that one side of the equation would contain only the term in which o is raised to its highest power. From this, we will be able to determine the maximum, which is the aim of this technique. This transposition gives us

$$ba - 2a^2 + 2oa = o^2.$$

But, by hypothesis, o is the maximum quantity; therefore o^2, the square of the maximum quantity, must itself be maximum. Consequently, $ba - 2a^2 - 2oa$ (the expression equal to o^2) will be a maximum. However, it contains no radical. Following the method, let us treat it as if o were a known quantity. We will have the *coequality*

$$ba - 2a^2 + 2oa \sim ba + be - 2a^2 - 2e^2 - 4ae + 2oa + 2oe.$$

Let us eliminate the common terms, and divide the others by e:

$$b + 2o \sim 2e + 4a.$$

Removing $2e$ according to rule, we will have

$$b + 2o = 4a, \qquad \text{whence} \qquad 4a - b = 2o \qquad \text{or} \qquad 2a - \tfrac{1}{2}b = o.$$

Having established this equation by the method, we must return to the first, in which we had supposed $a + \sqrt{(ba - a^2)} = o$.

But we have just found that $o = 2a - \tfrac{1}{2}b$; therefore

$$2a - \tfrac{1}{2}b = a + \sqrt{(ba - a^2)}, \qquad \text{whence} \qquad a - \tfrac{1}{2}b = \sqrt{(ba - a^2)}.$$

Squaring:

$$a^2 + \tfrac{1}{4}b^2 - ba = ba - a^2,$$

whence, finally

$$ba - a^2 = 1/8\ b^2;$$

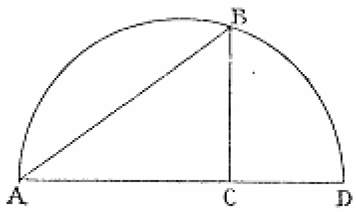
from this last equation we may draw out the value of a corresponding to the sought-for maximum.

We can employ this same technique to find *the cone with maximum surface which can be inscribed within a given sphere*.

Let AD (*fig.* 99) be the diameter of this sphere, AC the height of the sought-for cone, AB its side, and BC the radius of its base. According to Archimedes, it is necessary that the sum AB×BC+BC^2 be a maximum.

[6] Yes, this is the vowel o. In the Œuvres, a macron is placed above the $\bar{o}$ to avoid confusion with zero.

Fig. 99.



Let the diameter be b, and let AC = a. We will have AB = $\sqrt{ba}$, and BC=$\sqrt{(ba - a^2)}$,

$$AB \times BC + BC^2 = \sqrt{(b^2a^2 - ba^3)} + ba - a^2.$$

Let us equate this sum with the maximum area, o.

$$o + a^2 - ba = \sqrt{(b^2a^2 - ba^3)}.$$

Now we square, etc. The indicated method brings us to an equation that gives o and thus permits us to resolve that which we have posed.

However, in the chosen example, we can obtain the solution without using a third unknown; for we can reduce the problem as follows: given the line AB in the triangle CBA, find the maximum of the ratio $(CB \times BA + CB^2)/AD^2$. In this case, the ordinary method is sufficient.

Let the given line AB be represented by b, and let CB = a. We will have $AC^2 = b^2 - a^2$. But $AC^2/AB^2 = AB^2/AD^2$; therefore $AD^2 = b^4/(b^2 - a^2)$. Yet we desire that the ratio of $ba + a^2$ to this last expression be maximum.

Multiply the top and bottom by $b^2 - a^2$; the ratio $b^4/(b^3a + b^2a^2 - ba^3 - a^4)$ must be minimum. But b^4 is given, as a power of the given b; therefore the quantity $b^3a + b^2a^2 - ba^3 - a^4$ must be maximal.

The method gives the equation

$$b^3 + 2b^2a = 3ba^2 + 4a^3,$$

whose degree can be immediately reduced:[7]

$$4a^2 - ba = b^2;$$

from which the solution is clear.

We will not linger any longer on a subject henceforth elucidated; we see how, by making use of a third or a fourth unknown, and, if necessary, by again multiplying the number of auxiliary positions, we may rid ourselves of radicals and all the other obstacles which can hold up analysis.

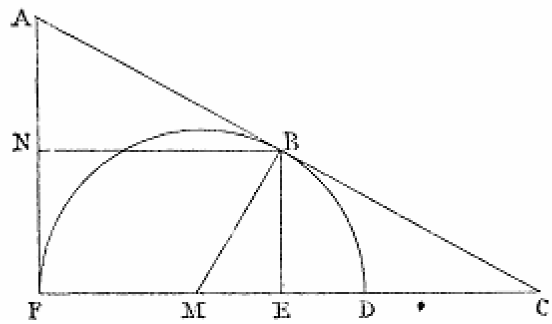
However, although the invention of tangents proceeds from the general method, we may remark that, in certain cases, the question of maxima and minima may be solved more elegantly and perhaps more geometrically, by means of constructing a tangent.

Let us give one example, which can count for several:

In a semicircle FBD (*fig.* 100), *draw the perpendicular* BE*; we seek the maximum of the product* FE×EB.

[7] By taking into account the root $a = - b$. (Note from the editors of the Œuvres)

Fig. 100.



If, according to our method, we seek to construct the rectangle FE×EB by giving it a value, the question comes down to describing a hyperbola having AF and FC as its asymptotes and where the product of its abscissas FE and ordinates EB should have that given value; the points of intersection of the hyperbola and the semicircle will fulfill the question. But, since the product FE×EB must be a maximum, we must, in fact, make a hyperbola which has AF and FC as its asymptotes, and which, instead of intersecting the semicircle, is tangent to it instead, at B. For the points of contact determine maximum and minimum quantities.

Let us suppose the problem solved: if the hyperbola touches the semicircle at B, the tangent to the semicircle at B will also be tangent to the hyperbola. Let this line be ABC. It is tangent to the hyperbola at B and touches the asymptotes at A and C; therefore, according to Apollonius, AB = BC. Consequently, FE = EC and AF = 2BE = 2AN. But, since it is tangent to the circle, BA = AF; therefore BA = 2AN, and by the similarity of triangles, if M is the center, MB = 2ME. But the radius MB is given; therefore the point E will be known.

Similarly, we can in general reduce any search for a maximum or a minimum to the geometric construction of a tangent; but this does not in any way diminish the importance of the general method, since the construction of tangents depends on it, just as the determination of maxima and minima.

VI
On the Same Method

The theory of tangents is a result of the long-published method for the finding of maxima and minima, which permits the easy solution of all problems of limits, and notably the famous problems whose limit-conditions were considered difficult by Pappus (Book VII, preface).

The curved lines for which we are seeking tangents can be expressed by their specific properties, either by means of straight lines alone, or by means of a mixture of complex curves, as one wishes, with the help of straight lines or other curves.

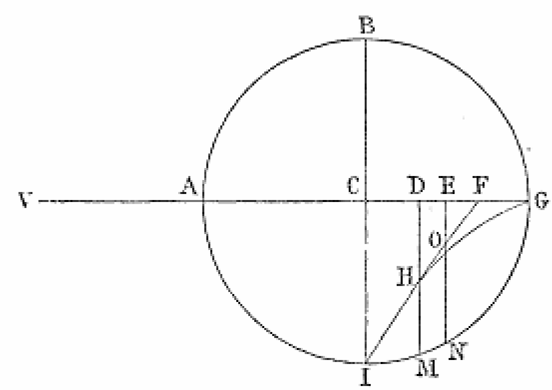
With our rule, we have already met the conditions for the first case which may have appeared to be difficult because it was too concise; however, it has nonetheless been recognized as legitimate.

In fact, in the plane of an arbitrary curve, we consider two lines given in position, one called the *diameter*, the other, the *ordinate*. We assume the tangent to be already found at a given point on the curve, and we consider by *ad-equality* the specific property of the curve, no longer on the curve itself, but on the sought-for tangent. Following our theory of maxima and minima, we eliminate those terms which ought to be eliminated, and arrive at an equality which determines the point of intersection of the tangent with the diameter, and then later the tangent itself.

To the numerous examples which I have already given, I will add that of the tangent to the *cissoid*, invented, it is said, by Diocles.

Let there be a circle wherein two diameters AG, BI (*fig.* 101) cut each other perpendicularly, and let there be cissoid IHG, on which, through any of its points, say H, we must draw the tangent.

Fig. 101.



Let us consider the problem to be solved, and suppose F to be the intersection of CG and the tangent HF. Let us call DF = a, and, in taking an arbitrary point E between D and F, let us say that DE = e.

Making use of the property specific to the cissoid that MD/DG = DG/DH, we will thus have to express analytically the *adequality* NE/EG ~ EG/EO, EO being the portion of the line EN between E and the tangent.

Let AD = z be given, DG = n be given, DH = r be given, and, as we have said, the unknown DF = a, the arbitrary DE = e.

We will have

$$EG = n - e, \qquad EO = (ra - re)/a, \qquad EN = \sqrt{(zn - ze + ne - e^2)}.$$

According to rule, we must consider the specific property, not on the curve, but on the tangent, and therefore state that NE/EG=EG/GO, EO being the ordinate of the tangent. Or, in analytical notation:

$$\sqrt{(zn - ze + ne - e^2)}/(n - e) \sim (n - e)/[(ra - re)/a].$$

Squaring, to remove the radical:

$$(zn - ze + ne - e^2)/(n^2 + e^2 - 2ne) \sim (n^2 + e^2 - 2ne)/[(r^2a^2 + r^2e^2 - 2r^2ae)/a^2].$$

Multiplying all the terms by a^2, and, according to rule, *adequalizing* the product of the extremes with the square of the mean, and removing the superfluous terms, pursuant to our method, we will finally have

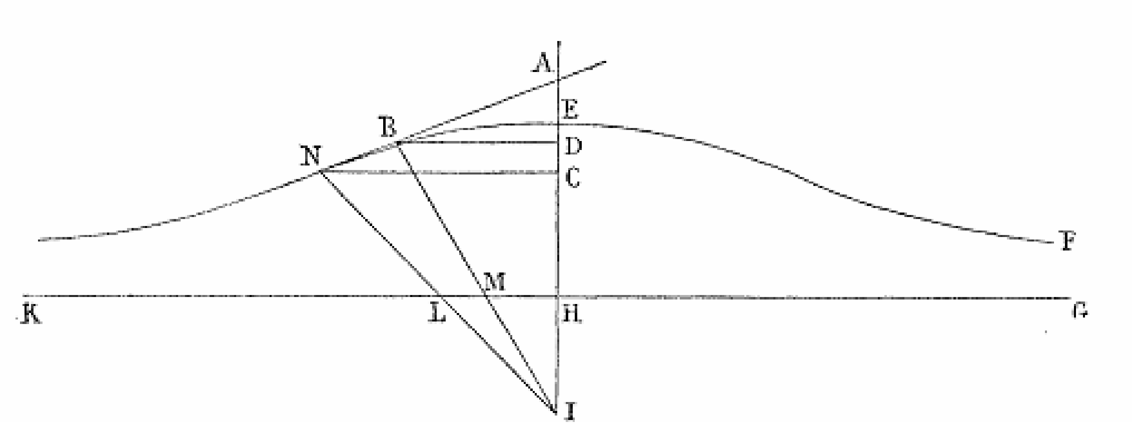
$$3za + na = 2zn.$$

Whence comes the following construction of the tangent: Extend radius CA to V such that AV = AC. Divide AD×DG by VD, make DF the quotient; connect FH. You will have the tangent of the cissoid.

We also indicate how to proceed for the *conchoid of Nicomedes*, but we will only sketch it out so as not to be too prolix.

Let the conchoid of Nicomedes be constructed in the figure as it is in Pappus and Eutocius (*fig.* 102). The pole is I, KG the asymptote, IHE the perpendicular to the asymptote, N a given point on the curve, through which we are to draw a tangent NBA meeting IE at A.

Fig. 102.



Let us suppose the problem solved, as above. Let us draw NC parallel to KG. By the property specific to the curve, we have LN = HE. Let us take an arbitrary point between C and E, say D, and draw DB through this point, parallel to CN, letting DB reach the tangent at B. Since the property specific to the curve must be considered on the tangent, let us join BI which meets KG at M. Following the rules of the art, we must ad-equate MB and HE. We will thus arrive at the sought-for equation. For this, we will say, as above, that CA = a, CD = e, EH = z, and we will call the other ordinates by their names. We will easily find the analytical expression of the line MB. We will then *adequalize* it, as has been said, to line HE, and solve the problem.

What I have said seems to suffice for the first case. It is true that there are an infinite number of artifices to shorten calculations in practice; but one can easily deduce them from what has come before.

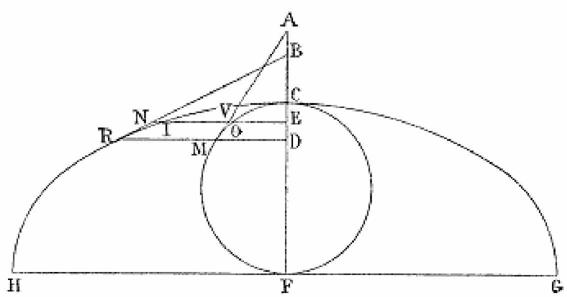
As for the second case that M. Descartes considered difficult, and for whom nothing is, we have brought it to satisfaction by way of a very elegant and subtle method.

As long as the terms are formed only by straight lines, we may find and designate them according to the preceding rule. Moreover, to avoid radicals, we may substitute, in place of the ordinates to the curve, the ordinates of the tangents found by the preceding method. Finally, and most importantly, we may substitute, for the arcs of the curves, the corresponding lengths of the already-found tangents, and arrive at the *adequality*, as we have indicated: thus we will easily satisfy the question.

Let us take the curve of M. de Roberval – the cycloid – as an example.

Let HBIC be the curve (*fig.* 103), C its summit, and CF the axis; let us describe the semicircle COMF, and take an arbitrary point on the curve, say R, from which the tangent RB is to be drawn.

Fig. 103.



Let us draw from this point R, perpendicularly to CDF, the line RMD, cutting the semicircle at M. The property specific to this curve is that the line RD is equal to the sum of the arc of the circle CM and the ordinate DM. Let us then draw, following our preceding method, MA, the tangent to the circle (the same process would indeed be applicable if the curve COM were of another nature). Let us suppose the construction performed, and let the unknown DB = a, the lines found by construction: DA = b, MA = d; the givens MD = r, RD = z, the given arc of the circle CM = n, and the arbitrary line DE = e.

From E draw EOVIN parallel to the line RMD; we have $a/(a - e) = z/$NIVOE, whence we have NIVOE = $(za - ze)/a$.

Therefore it is necessary to *ad-equate* (because of the specific property of the curve which is to be considered on the tangent) this line $(za - ze)/a$ to the sum OE + arcCO.

But arcCO = arcCM – arcMO. Therefore $(za\text{-}ze)/a \sim$ OE + arcCM – arcMO.

To obtain the analytic expression of the three last terms, while avoiding radicals, we can, according to the preceding remark, substitute the ordinate of the tangent EV for OE, and the portion of the tangent MV for arc MO.

To find the analytic expression of EV, we moreover have $b/(b - e)=r/$EV, whence EV = $(rb - re)/b$.

For MV, by reason of similar triangles, we have, as above, $b/d = e/$MV, whence MV = de/b.

Finally we have arcCM = n. Thus, we have, analytically:

$$(za\text{-}ze)/a \sim (rb - re)/b + \mathrm{n} - de/b.$$

Multiplying both sides by ab:

$$zba - zbe \sim rba - rae + bna - dae.$$

But, from the property of the curve, we know that $z = r + n$, and therefore $zba = rba + bna$.

Removing the common terms, we have

$$zbe \sim rae + dae.$$

Let us divide by e; since there are no more superfluous terms here, there are no other eliminations to make:

$$zb = ra + da, \qquad \text{whence} \qquad (r + d)/b = z/a.$$

For the construction, we will then make (MA + MD)/DA = RD/DB; we will join BR which will touch curve CR.

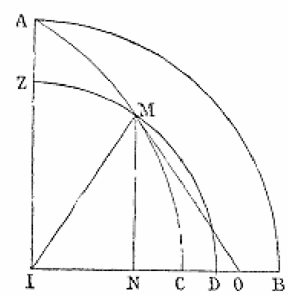
But since (MA+MD)/DA=MD/DC, as is easy to demonstrate, we can have MD/DC = RD/DB, or, to make the construction more elegant, we may join MC and then draw RB parallel.

The same method will give the tangents to all curves of this type. We have indicated their general construction a long time ago.

Since it has been proposed to find the *tangent of the quadratrix of Dinostratus*, here is how we can construct it according to the preceding method.

Let AIB be a quarter of a circle (*fig.* 104), AMC the quadratrix, from which we must draw the tangent at a given point M. I join MI, and then with I as the center and IM the radius, I draw the quarter circle ZMD. Drawing the perpendicular MN, I make MN/IM = arcMD/IO. I join MO which will be tangent to the quadratrix; this should be sufficient.

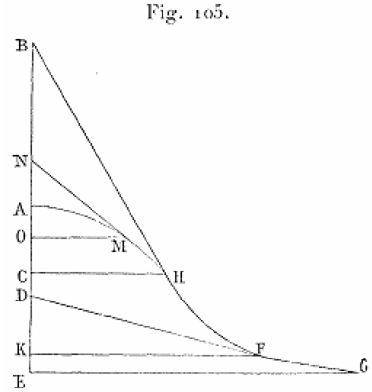
Fig. 104.



However, it often occurs that the curvature changes, as for the conchoids of Nicodemus (1st case) and for all species except for the first one, the curve of M. de Roberval (2nd case). To be able to draw the curve well, it is suitable therefore to mathematically research the points of inflection, where the curvature changes from convex to concave, or the inverse. This question can be elegantly resolved by the method of *maximum and minimum*, thanks to the following general lemma:

Let there be, for example, the curve AHFG (*fig.* 105) *whose curvature changes at point* H. *Draw the tangent* HB *and the ordinate* HC; *the angle* HBC *will be the minimum among all those angles which the tangent makes with axis* ACD *when it be below or above point* H, as is easy to demonstrate.

Fig. 105.



Let us take point M above point H. The tangent to this point will reach the axis at a point between A and B – let it be N. The angle at N will therefore be greater than the angle at B.

Similarly, if we take the point F below H, then the point D where the tangent DF meets the axis will be below B. Moreover, the tangent DF will meet the tangent BH on the side of FH. The angle at D will therefore be greater than the angle at B.

We will not pursue all cases, preferring only to indicate the mode of study, since the forms of curves vary infinitely.

Therefore to find, for example, the point H on the shape, we will first seek, following the preceding method, the property of the tangent at an arbitrary point of the curve. Since, by the doctrine *of maxima and minima*, we will determine the point H such that by drawing the perpendicular HC and the tangent HB, the ratio HC/CB will be a minimum. For thus the angle at B will be a minimum. I say that the point H thus determined will be that where the change in curvature is found.

The same method *of maxima and minima* gives also, by a singular expedient, the determination of the center of gravity, as I have indicated to M. de Roberval in the past.

But, as a crowning achievement, we can even *find the asymptotes of a given curve*, a study which leads to remarkable properties for indefinite curves. We shall develop and demonstrate them more at length at some future time.

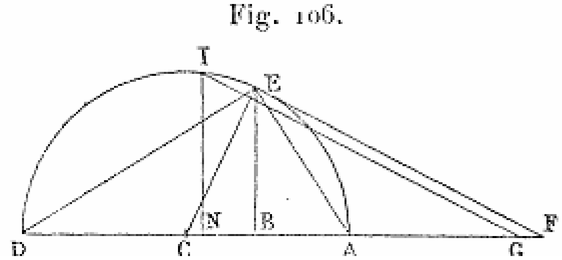
VII
Problem Sent to the Rev. Father Mersenne

on the 10th of November, 1642

Find the cylinder of maximum surface area inscribed in a given sphere.

Let there be given a sphere of diameter AD (*fig.* 106), with center C. It is demanded to inscribe within it the cylinder of maximum surface.

Fig. 106.



Let us suppose the problem solved. Let DE be the diameter of the base of the cylinder, EA its side (we can indeed give this position on the cylinder, the angle inscribed in the semicircle being right). The surface of the cylinder is proportional to $DE^2 + 2DE \cdot EA$; it is therefore necessary to find the maximum of the sum $DE^2 + 2DE \cdot EA$.

If the perpendicular EB be dropped, we have for one term $DE^2 = AD \cdot DB$, and for the other $DE \cdot EA = AD \cdot BE$. Thus, we must find the maximum of the sum $AD \cdot DB + 2AD \cdot BE$, or, by dividing the terms by the given line AD, the maximum of the sum $DB + 2BE$.

This question is easy. If we let CB = ½BE, or (which is really the same thing), BC = CE/√5, the point E will solve the problem. Indeed, draw the tangent EF which reaches the extension of the diameter at F. I say that the sum DB + 2BE is maximal.

Since CB=½BE, BE=½BF; therefore BF = 2BE; therefore DF=DB+2BE. Thus, it is clear that the sum DB+2BE is maximal.

Indeed, let us take an arbitrary point on the semicircle – let it be I – and from it let us drop the perpendicular IN; from the same point I, draw IG parallel to the tangent and intersecting the diameter at G. This point G will fall between the points F and D, for otherwise the parallel GI could not reach the semicircle. By reason of parallels, we have FB/BE=GN/NI. But FB=2BE; therefore GN=2NI, and GD=DN+2NI. But since GD(=DN+2NI) is smaller than DF(=DB+2BE), it follows that DB+2BE is a maximum and that the desired cylinder will have DE for its base and EA for its side.

It can be proven, following the preceding, that the ratio DE/EA is that of the greatest to the smallest segments of a line divided in the mean and extreme ratio.[8]

Moreover, by the same process, we may *find and construct a cylinder with a given surface area.*

This problem will be reduced to the question of the equality of the sum DN+2NI and a given line, that is, DG, which, after having found the value for the maximum, should be at most equal to DF. Draw GI parallel to FE; the point I will solve the problem, and one could have two cylinders just as well as one in the proposed condition.

If, indeed, point G falls between F and A, two different cylinders will satisfy the problem; but if G falls on A or any point closer to D, the solution will be unique.

VIII
Analysis for Refraction[9]

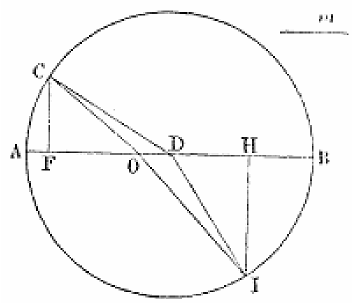
Let ACBI (*fig.* 108) be a circle whose diameter AFDB separates two media of different natures, the less dense being on the side ACB, and the more dense being on the side AIB.

Let D be the center of the circle, and let CD be an incident ray falling upon the center from given point C; we seek to know the refracted ray DI, or point I through which the ray passes after refraction.

[8] Also known as the golden ratio, or the divine proportion.

[9] This piece was sent by Fermat to M. de la Chambre, accompanying his letter of Jan 1, 1662.

Fig. 108.

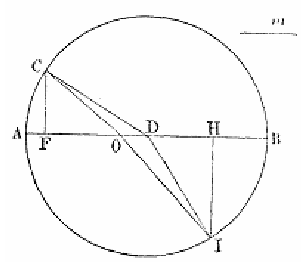


Drop perpendiculars CF and IH onto the diameter. Point C being given, along with the diameter AB and the center D, the point F and the line FD are also known. Let us suppose that the ratio of the resistance of the denser medium to the resistance of the rarer medium, be the same ratio as that between given line DF and another given line m drawn outside the figure. We will have $m <$ DF, the resistance of the rarer medium necessarily being smaller than that of the denser, by a more than natural axiom.

By means of the lines m and DF, we now have to measure the movements along lines CD and DI; we will thus be able to comparatively represent the entirety of the movement along these two lines by the sum of two products: $CD{\cdot}m + DI{\cdot}DF$.

Thus the question comes down to the division of the diameter AB at a point H such that if a perpendicular HI be erected at this point, and DI be joined, the area $CD{\cdot}m + DI{\cdot}DF$ will be a minimum.

Fig. 108.



To this effect, we will employ our method, already widespread among geometers and exposed about twenty years ago by Hérigone in his *Cursus mathematicus*. Let us call n the radius CD or its equal DI, b the line DF, and DH = a. The quantity $nm+nb$ must be a minimum.

Let us take an arbitrary line DO as our unknown e, and join CO and OI. In analytical notation, $CO^2 = n^2 + e^2 - 2be$, and $OI^2 = n^2 + e^2 + 2ae$; therefore

$$CO{\cdot}m = \sqrt{(m^2n^2+m^2e^2 - 2m^2be)},$$

$$IO{\cdot}b = \sqrt{(b^2n^2+b^2e^2+2b^2ae)}.$$

Following the rules of the art, the sum of these two radicals must be *ad-equated* to $mn+bn$.

To cause the radicals to disappear, we will square them, and we will do away with the common terms. We will transpose the terms in such a way as to leave only the remaining radical on one of the sides of the equation. Then we will square the new equation. After new eliminations of terms on both sides, division by e and removal of terms containing e, following the rules of our method generally known for a long time, then by removing common factors, we will arrive at the simplest possible equation involving *a* and *m*. That is to say that after having removed the obstacle of the radicals, we will find that the line DH of the figure is equal to the line *m*.

Consequently, after having drawn the lines CD and DF, we may find the point of refraction by taking the lines DF and DH to be in the ratio of the resistance of the denser to the rarer medium – the ratio of *b* to *m*. From H we will then erect line HI perpendicular to the diameter; it will intersect the circle at I, the point where the refracted ray will pass. And thus the ray, passing from a rarer to a denser medium, will bend away from the side and towards the perpendicular: which agrees absolutely and without exception with the theorem discovered by Descartes. The above analysis, derived from our principle, therefore gives this Cartesian theorem a rigorously exact demonstration.

IX
Synthesis for Refractions[10]

Descartes the savant proposed a law for refractions which, as they say, conforms with experience; but to demonstrate it, he had to rely on a postulate absolutely indispensable to his reasoning, namely: that light moves more easily and more quickly in dense media than in rare. Yet this postulate seems to be contrary to the light of reason.

In seeking to establish the true law of refraction by starting from the contrary principle – namely, that the movement of light is easier and quicker in rare than in dense media – we come upon the precise law that Descartes had enunciated. Without paralogism, is it possible to arrive at the same truth by two absolutely opposite paths? This is a question that we will leave to the scrutiny of geometers who have sufficient shrewdness to resolve it rigorously, because, without entering into vain discussions, the assured possession of truth is sufficient for us and we consider it preferable to a long series of useless and illusory quarrels.

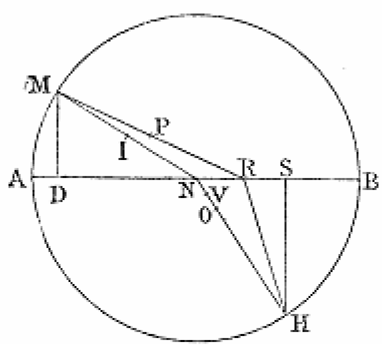
Our demonstration relies solely upon the postulate that nature operates by the easiest and most convenient means and pathways. For we believe it must be stated this way, and not in the ordinary way, which says that nature always operates by the shortest lines.

Indeed, in addition to speculating on the natural movements of heavy bodies, Galileo measured their relationships in time as well as in space, similarly, we will not consider the shortest spaces or lines, but rather those pathways which can be most easily traveled through with the greatest of ease, in the most accommodating way, and in the least time.

This posed, let there be two media of different natures (*fig.* 109) separated by the diameter ANB of a circle AHBM, the less dense medium being on the side of M, and the more dense on the side of H.

[10] According to the copy of this piece that Clerselier received, Fermat sent it to Mersenne in February 1662.

Fig. 109.



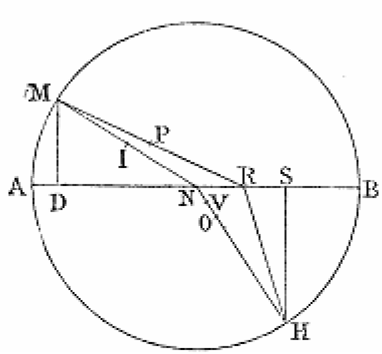
Draw from M towards H the arbitrary lines MNH, MRH, intersecting the diameter at points N and R.

According to the axiom or the postulate, the speed of the element propagating along MN in the rarer medium being greater than the speed of the same element moving along NH, and the motions being considered to be uniform within both media, the ratio between the time of the motion along MN and the time of motion along NH will be, as we know, the product of the ratio of MN to NH with the inverse ratio of the speeds along NH and along MN. Thus we have (speed along MN)/(speed along NH) = MN/NI, and we will have (time through MN)/(time through NH) = IN/NH.

We will likewise prove that if the ratio of speeds in the rare and dense media is MR/RP, we will have (time through MR)/(time through RH) = PR/RH. Whence it follows that (time through MNH)/(time through MRH) = (IN + NH)/(PR + RH).

Now, since it is nature that directs light from point M to point H, we must find point N, through which the light passes in the least time from point M to point H as it bends or refracts. For we must admit that nature, which directs its motions as quickly as possible, aims for this point by itself. If, therefore, the sum IN + NH, which measures the time of motion along bent line MNH, is a minimal quantity, we will have attained our goal.

Fig. 109.



The statement of the theorem of Descartes gives this minimum, as we will soon prove by a true geometric reasoning and without any ambiguity. Here indeed, is the exposition:

If from point M we draw the radius MN, and from the same point M drop the perpendicular MD, and if we take DN/NS as the ratio of the greater speed to the lesser, and, finally, if we erect from S the perpendicular SH and draw the radius NH, the incident light at point N in the rare medium will refract into the dense medium, moving towards the perpendicular, and arriving at point H.

It is this theorem which is in accord with our Geometry, since it results from the following purely geometric proposition.

Let there be circle AHBM, with diameter ANB and center N. On the circumference of this circle, take an arbitrary point M, draw the radius MN, and drop perpendicular MD onto the diameter. Let the ratio DN/NS also be given, supposing DN > NS. From S, erect perpendicular SH from the diameter, which reaches the circumference at point H. Join this point to the center by radius HN. Suppose that DN/NS = MN/NI; I then say that the sum IN + NH is a minimum. That is to say that if we took another point on line NB, such as R, joined MR and RH, and made DN/NS = MR/RP, we would find that PR + RH > IN + NH.

To demonstrate this, let MN/DN = RN/NO and DN/NS = NO/NV. By construction, it is clear that since DN is smaller than the radius MN, it must be that NO < NR. Likewise, since NS < ND, we will conclude that NV < NO.

This posed, we have, according to Euclid, $MR^2 = MN^2 + NR^2 + 2\ DN.NR$. But since by construction we have MN/DN = NR/NO, we say that MN·NO = DN·NR, and therefore 2MN·NO = 2DN.NR. Therefore $MR^2 = MN^2 + NR^2 + 2MN \cdot NO$.

Now, since NR > NO, $NR^2 > NO^2$. Therefore

$$MR^2 > MN^2 + NO^2 + 2MN \cdot NO.$$

But the sum $MN^2 + NO^2 + 2MN \cdot NO = (MN + NO)^2$. Therefore

$$MR > MN + NO.$$

On the other hand, we have by construction, DN/NS = MN/NI = NO/NV, and therefore

$$DN/NS = (MN+NO)/(IN+NV).$$

But we also have DN/NS = MR/RP. Therefore (MN+NO)/(IN+NV)=MR/RP. Yet MR>MN+NO; therefore it is also true that RP>IN+NV.

It remains to be proven that RH>HV; for, if it is so, it is clear that PR+RH > IN+NH.

Now in triangle NHR, according to Euclid,

$$RH^2 = HN^2 + NR^2 - 2SN.NR.$$

But by construction (MN(=NH))/DN = NR/NO, and DN/NS = NO/NV; therefore, *ex æquo*, HN/NS = NR/NV. Therefore HN·NV = NS·NR and 2HN·NV = 2SN·NR. Therefore

$$RH^2 = HN^2 + NR^2 - 2HN \cdot NV.$$

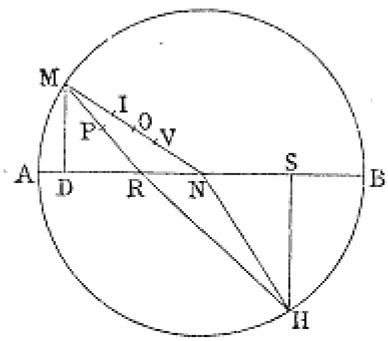
But we have proven that $NR^2 > NV^2$. Therefore

$$HR^2 > HN^2 + NV^2 - 2HN \cdot NV.$$

Then $HN^2 + NV^2 - 2HN \cdot NV = HV^2$, according to Euclid; therefore $HR^2 > HV^2$ and HR > HV, which remained to be proven.

Even if we take point R on radius AN, such that lines MR and RH were to extend into each other, as in the following figure (*fig.* 110) – although the demonstration is independent of this particular case – the result would be the same, which is to say that we will always have PR + RH > IN + NH.

Fig. 110.



Let us take, as above, $MN/DN = RN/NO$ and $DN/NS=NO/NV$. It is clear that $RN>NO$ and $NO>NV$.

$MR^2 = MN^2 + NR^2 - 2DN{\cdot}NR$. Following the above reasoning, we may substitute $2MN{\cdot}NO$ for $2DN{\cdot}NR$. Moreover $NR^2 > NO^2$; therefore $MR^2 > MN^2 + NO^2 - 2MN{\cdot}NO$. But

$$MN^2 + NO^2 - 2MN{\cdot}NO = MO^2.$$

Therefore $MR^2 > MO^2$ and $MR > MO$.

Additionally, we have by construction that $DN/NS=MN/IN=NO/NV$; therefore *vicissim:* $MN/NO=NI/NV$, and *dividendo:* $MO/ON=IV/VN$, and *vicissim:*

$$MO/IV = ON/NV = DN/NS = MR/RP.$$

But we have proven that $MR > MO$; therefore $PR > IV$. To establish the proposition, there remains to be proved that $RH > HN + NV$; which is quite easy after the preceding.

Indeed $RH^2 = HN^2 + NR^2 + 2SN{\cdot}NR$; as we have seen, we can substitute $2HN{\cdot}NV$ for $2SN{\cdot}NR$; moreover we have $NR^2 > NV^2$. Therefore

$$HR^2 > HN^2 + NV^2 + 2HN{\cdot}NV;$$

Therefore, as above, $HR > HN + NV$.

It is therefore certain that the sum of the two lines PR and RH, even when they form a straight line PRH, is always greater to the sum $IN + NH$. Q.E.D.

END
