On the Transformation

and the

Simplification of Equations of Loci,

for the Comparison under all forms
of Curvilinear Areas, be it among them, be it with Rectilinear Areas,
and at the Same Time
ON THE USE OF THE GEOMETRIC PROGRESSION

for the Quadrature of Parabolas and Hyperbolas to Infinity

Pierre de Fermat

Archimedes used the geometric progression only for the quadrature of the parabola. In his other comparisons between heterogeneous quantities, he limited himself to the arithmetic progression. Is it because he found that the geometeric progression was less suited to quadrature? Is it because the particular technique that he used to square the first parabola with this progression was difficult to apply to others? Whatever the case may be, I recognized and found this progression to be quite fruitful for quadratures, and I gladly communicate my invention to modern geometers, an invention which permits squaring both parabolas and hyperbolas, by an absolutely identical method.

This whole method derives from a single well-known property of geometric progressions, namely the following theorem:

Given a geometric progression whose terms decrease indefinitely, the difference between two adjacent terms of this progression is to the smallest of the two as the greatest of all the terms is to the sum of all the others up to infinity.[1]

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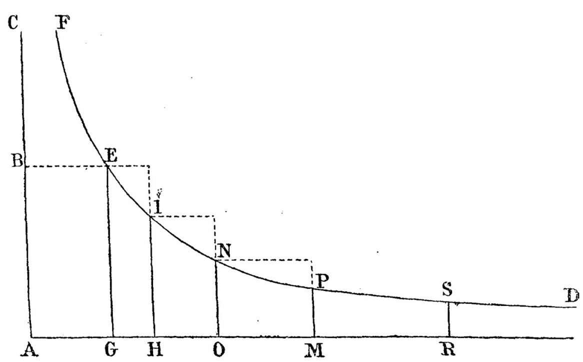
[1] Here is an example: The sum of all the terms of the geometric progression $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ except for 3 is $1\frac{1}{2}$. The ratio of 3 to $1\frac{1}{2}$ is a ratio of 2 to 1. Similarly, for the two terms $\frac{1}{3}, \frac{1}{9}$, their difference is $\frac{2}{9}$, which is in a ratio of 2 to 1 with the smaller term, $\frac{1}{9}$.

This posed, let us first set out to find the quadrature of hyperbolas:

I define hyperbolas as the species of curves, infinite in number, which, like DSEF (*fig.* 142), have this property: if we assume, under an arbitrary given angle RAC, the asymptotes RA, AC, which go on infinitely like the curve itself, and if we draw the lines GE, HI, ON, MP, RS, etc., parallel to one of the asymptotes, we will always have the same ratio between a determined power of AH and the same power of AG on one side, and a power of GE (similar or different to the preceding ratio) and the same power of HI, on the other side. By powers, I mean not only squares, cubes, biquadratics, etc., with exponents 2, 3, 4, etc., but also simple roots whose exponent is 1.

Fig. 142.



I say that all these infinite number of hyperbolas, excepting only one – that of Apollonius, or the primary hyperbola – can be squared by means of a geometric progression, by a uniform and constant method.

Let there be, for example, the hyperbola whose characteristic is defined by the constant equality of the ratios $\frac{\text{HA}^2}{\text{AG}^2} = \frac{\text{GE}}{\text{HI}}$ and $\frac{\text{OA}^2}{\text{AH}^2} = \frac{\text{HI}}{\text{ON}}$, etc. I say that the space which has GE for its base and which is limited on one side by the curve ES, and on the other by the infinite asymptote GOR, is equal to a given rectilinear area.

Let us imagine the terms of a geometric progression which is always decreasing. Let AG be the first term, AH the second term, AO the third term, etc. Let us assume that these terms are close enough to each other so that we may, following the method of Archimedes, *ad-equate* them (as Diophantus says), or equate by approximation the rectilinear parallelogram GE $\times$ GH to the curvilinear quadrilateral GHIE; we will additionally assume that the first intervals GH, HO, OM, etc. of the progressing terms are sufficiently equal among themselves, for us to be able to easily employ the method of

Archimedes of reduction to the impossible,[2] by circumscriptions and inscriptions. It suffices to make this remark once so that we may not be required to return to it repeatedly, making too much of a technique that is well known to all geometers.

Having posed this, since $\frac{\mathrm{AG}}{\mathrm{AH}} = \frac{\mathrm{AH}}{\mathrm{AO}} = \frac{\mathrm{AO}}{\mathrm{AM}}$, we will then have $\frac{\mathrm{AG}}{\mathrm{AH}} = \frac{\mathrm{GH}}{\mathrm{HO}} = \frac{\mathrm{HO}}{\mathrm{OM}}$, for the intervals. But, for the parallelograms, we will have:

$$\frac{\mathrm{EG}\times\mathrm{GH}}{\mathrm{HI}\times\mathrm{HO}} = \frac{\mathrm{HI}\times\mathrm{HO}}{\mathrm{NO}\times\mathrm{OM}}.$$

Indeed, the ratio of the parallelograms $\frac{\mathrm{GE}\times\mathrm{GH}}{\mathrm{HI}\times\mathrm{HO}}$ is composed of the ratios $\frac{\mathrm{GE}}{\mathrm{HI}}$ and $\frac{\mathrm{GH}}{\mathrm{HO}}$; but, as we have indicated, $\frac{\mathrm{GH}}{\mathrm{HO}} = \frac{\mathrm{AG}}{\mathrm{AH}}$; therefore the ratio $\frac{\mathrm{EG}\times\mathrm{GH}}{\mathrm{HI}\times\mathrm{HO}}$ is composed of the ratios $\frac{\mathrm{GE}}{\mathrm{HI}}$ and $\frac{\mathrm{AG}}{\mathrm{AH}}$. On the other hand, by construction, we have $\frac{\mathrm{GE}}{\mathrm{HI}} = \frac{\mathrm{HA}^2}{\mathrm{GA}^2}$ or $\frac{\mathrm{AO}}{\mathrm{GA}}$, as follows from the proportionality of terms. Therefore the ratio $\frac{\mathrm{EG}\times\mathrm{GH}}{\mathrm{HI}\times\mathrm{GO}}$ is composed of the ratios $\frac{\mathrm{AO}}{\mathrm{AG}}$ and $\frac{\mathrm{AG}}{\mathrm{AH}}$. But $\frac{\mathrm{AO}}{\mathrm{AH}}$ is composed of the same ratios. We will therefore have for the ratio of parallelograms $\frac{\mathrm{GE}\times\mathrm{GH}}{\mathrm{HI}\times\mathrm{HO}} = \frac{\mathrm{OA}}{\mathrm{AH}} = \frac{\mathrm{HA}}{\mathrm{AG}}$.

It can similarly be proved that $\frac{\mathrm{HI}\times\mathrm{HO}}{\mathrm{ON}\times\mathrm{OM}} = \frac{\mathrm{AO}}{\mathrm{HA}}$.

But the lines AO, HA, GA which constitute the ratios of the parallelograms, make, by construction, a geometric proportion; therefore the infinite number of parallelograms GE $\times$ GH, HI $\times$ HO, ON $\times$ NM, etc. will form a continuous geometric progression whose ratio will be $\frac{\mathrm{HA}}{\mathrm{AG}}$. Consequently, according to the theorem which makes up our method, the difference between the two terms of the ratio, GH, will be to the smaller term, GA, as the first term of the progression of parallelograms (i. e. parallelogram EG $\times$ GH), is to the sum of all the other infinite number of parallelograms, or, following the *ad-equation* of Archimedes, to the figure bounded by HI, asymptote HR, and the infinitely extended curve IND.

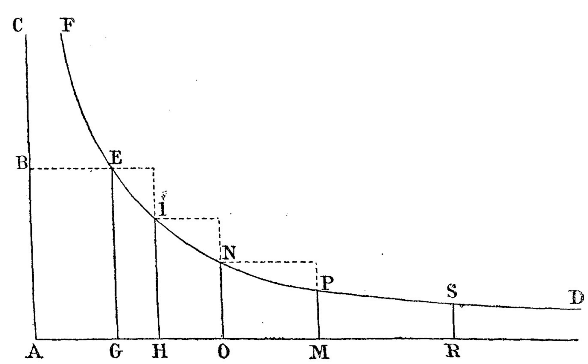
But, if we multiply these two terms by GE, $\frac{\mathrm{HG}}{\mathrm{GA}} = \frac{\mathrm{GE}\times\mathrm{GH}}{\mathrm{GE}\times\mathrm{GA}}$; therefore GE $\times$ GH is to this indefinite figure with base HI, as GE $\times$ GH is to GE $\times$ GA. Therefore the parallelogram GE $\times$ GA, which is a given rectilinear area, is *ad-equal* to the aforesaid figure; if we add the parallelogram GE $\times$ GH to both sides, which by following the indefinitely pursued divisions, will vanish and will reduce to nothing, we arrive to this truth which it would be easy to confirm by a more prolix demonstration, performed in the style of Archimedes: that in this type of hyperbola, the parallelogram AE is equivalent to the figure bounded by the base GE, asymptote GR, and the indefinitely extended curve ED.

[2] *ἀπαγωγὴν εἰς ἀδύνατον*

It is easy to extend this invention to all the hyperbolas, defined above, with the sole exception that we have indicated.[3] Indeed, let there be another hyperbola having for its property that $\frac{\text{GE}}{\text{HI}} = \frac{\text{HA}^3}{\text{GA}^3}$, etc. for the other ordinates.

Let us take, as above, an infinite series of geometrically progressing terms. We will similarly demonstrate that the parallelograms EH, IO, MN, etc. form a similar infinite progression; but in this case, the ratio of the first parallelogram to the second, of the second to the third, etc., will be $\frac{\text{AO}}{\text{GA}}$, which the composition of the ratios will immediately demonstrate. Therefore the parallelogram EH will be to the figure as OG is to GA, or, by multiplying the terms by GE, as OG $\times$ GE is to GE $\times$ GA: *vicissim* OG $\times$ GE is to EH or GE $\times$ GH as GE $\times$ GA is to the figure. But $\frac{\text{OG}\times\text{GE}}{\text{HG}\times\text{GE}} = \frac{\text{OG}}{\text{GH}}$ or $\frac{2}{1}$ by *ad-equation;*[4] for the neighboring intervals of the base are, by construction, sensibly equal among themselves. Therefore, in this hyperbola, the parallelogram EGA, which is equal to a given rectilinear area, will be twice the figure bounded by the base GE, asymptote GR, and the indefinitely extended curve ESD.

Fig. 142.



The demonstration will be the same in all the other cases; it is only for the primary hyperbola, that is to say the simple hyperbola or the hyperbola of Apollonius, that the method is deficient. The reason for this is that the parallelograms EH, IO, NM are always equal. The terms making up the progression, insofar as they are now equal to each other, give no difference, and it is precisely the difference which is the key to the whole matter.

I do not add the demonstration that, in the common hyperbola, the parallelograms in question are always equal. This is seen immediately and is

[3]The exception is the primary, or Apollonian hyperbola.

[4]Why is that?

instantly derived from this property of this type that always has $\frac{\text{GE}}{\text{HI}} = \frac{\text{HA}}{\text{GA}}$.

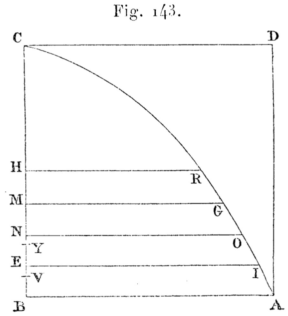
The same means can be used to square all the parabolas, except that again, as for the hyperbolas, there is one which escapes our method.

I will only give an example, that of the primary parabola, the parabola of Apollonius. On this model, we will be able to make all the demonstrations for any arbitrary parabola.

Let AGRC be a primary semi-parabola (*fig.* 143), of diameter CB and half-base AB. If we take the ordinates IE, ON, GM, etc., we always have $\frac{\text{AB}^2}{\text{IE}^2} = \frac{\text{BC}}{\text{CE}}, \frac{\text{IE}^2}{\text{ON}^2} = \frac{\text{EC}}{\text{CN}}$, and so on infinitely, according to the specific property of the parabola of Apollonius.

According to this method, let us imagine the lines BC, EC, NC, MC, HC, etc., forming an infinite progression. The parallelograms AE, IN, OM, GH will also form, as we have proven above, an indefinite progression.

Fig. 143.



To know the ratio of parallelograms AE, IN, we must, according to the method, return to the composition of the ratios..

Yet the ratio of parallelograms AE and IN is composed of the ratios $\frac{\text{AB}}{\text{IE}}$ and $\frac{\text{BE}}{\text{EN}}$. But, since $\frac{\text{AB}^2}{\text{IE}^2} = \frac{\text{BC}}{\text{CE}}$, if between BC and CE we take the mean proportional CV, and between EC and NC the mean proportional YC, then the lines BC, VC, EC, YC, NC will form a geometric progression, and we will have $\frac{\text{BC}}{\text{EC}} = \frac{\text{BC}^2}{\text{VC}^2}$; therefore, since $\frac{\text{BC}}{\text{EC}} = \frac{\text{AB}^2}{\text{EI}^2}$, $\frac{\text{AB}}{\text{IE}} = \frac{\text{BC}}{\text{VC}}$. Consequently, the ratio of parallelograms $\frac{\text{AE}}{\text{IN}}$ is composed of the ratio $\frac{\text{BC}}{\text{VC}}$ or $\frac{\text{VC}}{\text{CE}}$ or $\frac{\text{EC}}{\text{YC}}$ and of the ratio $\frac{\text{BE}}{\text{EN}}$ or, as we have proven above, $\frac{\text{BC}}{\text{EC}}$. But a ratio composed of the two ratios $\frac{\text{BC}}{\text{CE}}$ and $\frac{\text{EC}}{\text{CY}}$ is equal to the ratio $\frac{\text{BC}}{\text{CY}}$. Therefore the ratio between the parallelograms $\frac{\text{AE}}{\text{IN}} = \frac{\text{BC}}{\text{YC}}$, and consequently, according to theorem that constitutes our method, the ratio of parallelogram AE to the figure IRCHE is

$\frac{BY}{YC}$ and that of the same parallelogram AE to the total figure AIGRCB is $\frac{BY}{BC}$, where BC is the total diameter. But, if we multiply both sides by AB, then $\frac{BY}{BC} = \frac{AB \times BY}{AB \times BC}$; yet AB $\times$ BC is the parallelogram BD, obtained by drawing AD parallel to the diameter and by extending it until it intersects the tangent CD at D. Therefore the ratio of parallelogram AE to the semi-parabolic figure ARCB is the same as that of parallelograms AB $\times$ BY and BD. Therefore $\frac{AE}{AB \times BY} = \frac{ARCB}{BD}$. But since AE has AB as its edge, $\frac{AE}{AB \times BY} = \frac{BE}{BY}$; therefore $\frac{BE}{BY} = \frac{ARCB}{BD}$, or *convertendo,* $\frac{BD}{ARCD} = \frac{BY}{BE}$. But, because of the *adequality* of the lines BV, VE, EY, the differences between the terms of the progression, are assumed to be sensibly equal following the division into a very large number of very small parts, so $\frac{BY}{BE} = \frac{3}{2}$.[5] The ratio of parallelogram BD to the figure is therefore the ratio of 3 to 2, which is in accord with the quadrature of the parabola given by Archimedes, even though he used the geometric progression in a different way. If, moreover, I have found it necessary to change his method and to take a route other than his, it is because I am certain that by following the trail of this great geometer exactly, we would find the geometric progression to be useless for the quadrature of the infinite number of other parabolas, whereas our procedure immediately gives the demonstration and the general rules for all parabolas, without exception.

In order not to leave any doubts, let AIGC (*fig.* 144) be the parabola that I spoke of in my *Dissertation on the comparison of curved lines by straight lines,*[6] and let AB be its base, BC its diameter, IE, AB its ordinates such that we have $\frac{AB^3}{IE^3} = \frac{BC^2}{EC^2}$. Let us imagine the rest of the construction as above, that is to say, the indefinite progression of lines BC, EC, NC, MC, etc. and that of the parallelograms AE, IN, OM, etc.

Between BC and EC take the two mean proportionals VC, RC; similarly, between EC and CN, take the two mean proportionals SC, TC.

Since, by construction, $\frac{BC}{EC} = \frac{EC}{NC}$, the lines BC, VC, RC, EC, SC, TC, NC will be in a progression.

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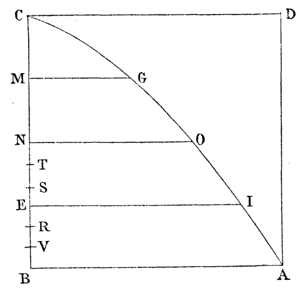
[5] Figure this one out too!

[6] Available at: `http://wlym.com/~animations/fermat`.

Fig. 144.



Moreover $\frac{AB^3}{IE^3} = \frac{BC^2}{EC^2} = \frac{BC}{NC}$. But, in the progression of the seven proportionals BC, VC, RC, EC, SC, TC, NC, the first, the third, the fifth, and the seventh also form a continuous progression.

Therefore BC : RC : SC : NC, and by taking the first, second, and fourth terms of this new progression, we have $\frac{BC}{NC} = \frac{BC^3}{RC^3}$; but we have proven that $\frac{BC}{NC} = \frac{AB^3}{IE^3}$; therefore $\frac{AB^3}{IE^3} = \frac{BC^3}{RC^3}$; whence $\frac{AB}{IE} = \frac{BC}{RC}$.

But the ratio between the parallelograms $\frac{AE}{IN} = \frac{AB}{IE} \times \frac{BE}{EN}$ or $\frac{BC}{RC} \times \frac{BC}{EC}$ (since also $\frac{BE}{EN} = \frac{BC}{EC}$).

On the other hand, in the seven terms in proportion, by taking the first, third, fourth, and sixth, we have $\frac{BC}{EC} = \frac{RC}{TC}$; therefore $\frac{AE}{IN} = \frac{BC}{RC} \times \frac{RC}{TC} = \frac{BC}{TC}$; therefore $\frac{AE}{IGCE} = \frac{BT}{TC}$.

Therefore, following what has been demonstrated, the ratio of the parallelogram to the figure is: $\frac{AE}{AICB} = \frac{BT}{BC} = \frac{AB \times BT}{AB \times BC}$, by multiplying both sides by AB; *vicissim* and *convertendo:* $\frac{BD}{AICB} = \frac{AB \times BT}{AB \times BE} = \frac{BT}{BE}$, by reason of the common side AB. But BT contains *five* intervals: TS, SE, ER, RV, VB which, by our logarithmic method, are considered to be equal to each other; and BE contains *three* interals: ER RV, VB: therefore, in this case, the ratio of the area of parallelogram BD to the area of the figure is the ratio of 5 to 3.

From this we can easily derive a universal rule: *It is indeed clear that the ratio of parallelogram BD to the figure AICB is always equal to the ratio of the sum of the exponents of the powers of the ordinate and the abscissa to the exponent of the power of the ordinate.* Thus, in this example, the power of the ordinate AB is cubic, of exponent 3, while that of the abscissa is square, of exponent 2. According to what we have established as a constant rule, we must have the ratio of the sum 3 + 2 = 5 to 3, the exponent of the ordinate.[7]

[7]Note that in his examples, Fermat has determined that the exponent of $\int x^{\frac{a}{b}}$ is $\frac{a+b}{b}$! (I'm not getting ahead of myself, am I?)

For hyperbolas, we can find a universal rule just as easily. *In an arbitrary hyperbola* (*fig.* 142) *the ratio of the parallelogram BG to the infinitely extended figure RGED will be equal to the ratio of: the difference between the exponent of the power of the ordinate and that of the abscissa, to the exponent of the power of the ordinate.* For example, let $\frac{\mathrm{HA}^3}{\mathrm{GA}^3} = \frac{\mathrm{GE}^2}{\mathrm{HI}^2}$; the difference between the exponents of the cube and the square, is $3 - 2 = 1$; the exponent of the power of the ordinate, which is squared, is 2. In this case the ratio of the parallelogram to the figure will be 1 to 2.

On the subject of the centers of gravity and tangents of hyperbolas and parabolas, their invention, derived by my *Method of Maxima and Minima,*[8] was communicated to modern geometers about twenty years ago. Without a doubt, the most celebrated mathematicians of France will want to make it known to foreigners, so that in the future there will be no doubt on this matter.

IT IS REMARKABLE how much the work of quadratures can be advanced by the preceding theory. For it permits us to easily square an infinite number of curves that both ancient and modern geometers had never thought of. We will briefly condense these results under the heading of certain rules.

Let there be a curve whose properties lead to the following equation:

$$b^2 - a^2 = e^2 \text{ (we immediately see that this curve is a circle).}$$

We can bring the power of the unknown e^2 to a root by a division (application or parabolism[9]). In fact, we may let

$$e^2 = bu;$$

for we are free to equate the product of the unknown u times the unknown b to the square of the unknown e. We will then have

$$b^2 - a^2 = bu.$$

But the term bu can be decomposed into as many terms as there are on the other side of the equation, while affecting these terms with the same signs as those on the other side. So let us take:

$$bu = bi - by,$$

[8]Available at: `http://wlym.com/~animations/fermat`.
[9]*parabolisme*

always representing unknowns by vowels, like Viète. There will result:

$$b^2 - a^2 = bi - by.$$

Let us equate each of the terms on one side of the equation with the corresponding term on the other side. We will have

$$b^2 = bi \quad \text{whence} \quad i = b \text{ will be given,}$$
$$-a^2 = -by \quad \text{or} \quad a^2 = by.$$

The extreme point of the line y will be on a primary (Apollonian) parabola. Thus, in this case, everything can be brought to a square; if we therefore set up all the values of e^2 as ordinates on a given straight line, their sum will be a known given rectangular solid.

Now let us examine the curve whose equation is:

$$a^3 + ba^2 = e^3.$$

Let us make e^3 a given area, for example: $e^3 = b^2u$.

Since line u can be composed of several different unknowns, let

$$a^3 + ba^2 = b^2i + b^2y.$$

Equate term by term, to whit:

$a^3 = b^2i$, we will have a parabola made of a cube and a root.
$ba^2 = b^2y$, we will have a parabola made of a square and a root, that is, a primary parabola.

Now, these two parabolas can be squared; therefore the sum of the e^3 set up as ordinates upon a straight line will form a *bi-plane*[10] that can easily be equated to rectilinear quantities of the same degree.

It there are a greater number of terms in the equation, or if the terms are composed of different powers of one or another of the unknowns, they can nonetheless be treated in an ordinary way by the same method, by use of legitimate reductions.

It is therefore clear that if in the first equation: $b^2 - a^2 = e^2$, we substitute bu for e^2, we can consider the sum of all the u values, ordered upon a straight

[10] *bi-plan*

line, as a *planar* figure, and square it.[11] Indeed, the sum of the u values is nothing other than the sum of the e^2 values, divided by a given line b.

Similarly, in the second question, the sum of the u values is nothing other than the sum of the e^3 values, divided by the given line b^2.

Therefore, in both the first and second cases, the sum of the u values makes a figure equal to a given rectilinear area.

These operations are made by *synérèse* and, as is clear, they are accomplished by means of parabolas.

But we obtain no fewer quadratures by *diérèse*, by means of hyperbolas, either alone, or combined with parabolas.[12]

For example, let us take up the curve having as its equation:

$$\frac{b^6+b^5a+a^6}{a^4} = e^2.$$

Similarly, we could say $e^2 = bu$, or rather, to have three terms on each side of the equation, we could say:[13]

$$bu = b\bar{o} + bi + by.$$

There will result

$$\frac{b^6+b^5a+a^6}{a^4} = b\bar{o} + bi + by. \text{ And, equating term by term:}$$

1^{st}: $\frac{b^6}{a^4} = b\bar{o}$; multiplying both sides by a^4 we have: $b^6 = a^4b\bar{o}$. Division by b gives: $b^5 = a^4\bar{o}$, the equation of a hyperbola. Indeed we know that the characteristic equations of hyperbolas include a given quantity on one side, and the product of powers of the two unknowns on the other.

2^{nd}: $\frac{b^5a}{a^4}$ or $\frac{b^5}{a^3} = bi$. Multiplying both sides by a^3 and dividing by b gives: $b^4 = a^3i$, the equation of a different hyperbola than the preceding.

3^{rd}: $\frac{a^6}{a^4}$ or $a^2 = by$, the equation of a parabola.

Therefore we see that, in the proposed equation, the sum of the u values set up as ordinates along a straight line, is equal to a given rectilinear area. For the sum of two squarable hyperbolas and a given parabola gives an area equal to a given rectangle or square.

[11] Does this not sound like the idea of integrating?

[12] *Synérèse* and *diérèse* are Viète's terms. Readers seeking to find out more about them may have luck using their Latin spellings: *synæresis* and *diæresis*.

[13] The bar over the $\bar{o}$ is used by Fermat to differentiate it from 0 (zero).

For the rest, nothing prevents us from separately dividing each of the terms of the numerator by the denominator, as we have done. Indeed, the result is the same as if we divided the entire three-term numerator by the denominator all at once. This separate division allows us to easily compare each term on one side of the equation with its correlated term on the other.

Now, let the following be proposed: $\frac{b^5 a - b^6}{a^3} = e^3$.

Let us say that $e^3 = b^2 u$, or, because of the two terms on the other side, $e^3 = b^2 i - b^2 y$. We will have:

1^{st}: $\frac{b^5 a}{a^3} = \frac{b^5}{a^2} = b^2 i$. Multiplying by a^2 and dividing by b^2, we have $b^3 = a^2 i$, the equation of a squarable hyperbola.

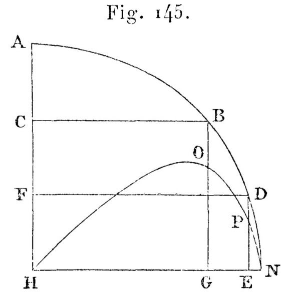
2^{nd}: $\frac{b^6}{a^3} = b^2 y$. Multiplying by a^3 and dividing by b^2, we have $b^4 = a^3 y$, an equation that constitutes a squarable hyperbola.

If we then return to the first equation, we will have, in this case, a given rectangle equal to the sum of all the e^3 values, set up as ordinates along a given line.

But nothing prevents us from going further in the work of quadratures.

Let there be an arbitrary curve ABDN (*fig.* 145), with base HN and diameter HA. Let CB, FD be ordinates on the diameter, and let BG, DE be ordinates on the base. We will assume that the ordinates decrease constantly from the base to the summit, as in the figure: that is, HN > FD, FD > CB, and so on and so forth.

Fig. 145.



The figure formed by the squares of HN, FD, CB, set up as ordinates upon the line AH, that is to say the solid $CB^2 \times CA \ldots + FD^2 \times FC \ldots + NH^2 \times HF$, is always equal to the figure formed by the rectangles $BG \times GH$, $DE \times EH$ doubled and the ordinates upon base HN — that is, to the solid 2 BG.GH.GH. . . + 2 DE.EH.EG, etc., the series of the terms on both sides being assumed to go on indefinitely. Yet, for other powers of the ordinates, the reduction of terms on the diameter to terms on the base is made with the same ease, and this observation leads to the quadrature of an infinite number of curves, unknown until now.[14]

Indeed, the sum of the cubes of HN, FD, CB, again ordinates on the line AH, will be equal to triple the sum of the products: BG.GH^2, DE.EH^2 and similarly ordered on the line HN, that is to say that the *bi-planar* CB^3.CA. . . + DF^3.FC. . . + HN^3.HF will be equal to the sum of the *bi-planars* 3(BG.GH^2.HG. . . + DE.EH^2.EG).

Similarly, the sum of the biquadratics of HN, FD, CB, set up as ordinates upon the line AH, will be equal to *four times* the the sum of the *bi-planars* BG.GH^3. . . DE.EH^3, set up as ordinates along line HN.

We will see that an infinity of quadratures can be derived from this.

For example, let us take curve ABDN, whose base HN and diameter HA are given. Let us analytically call by b the given diameter HA, and by d the given base HN. By e we will mean an arbitrary ordinate FD, and by a an arbitrary co-ordinate HF. And let us take, for example, $b^2 - a^2 = e^2$ as the characteristic equation of the curve (which will be a circle). According to the preceding general theorem, the sum of the values of e^2, as ordinates upon line b, is equal to the sum of the products HG.GB, doubled and set up as ordinates along the line HN or d. But the sum of the values of e^2, set up as ordinates on b is equal, as we have proved above, to a given rectangle. Therefore the sum of the products HG.GB, doubled and set up as ordinates on base d, will also form a given rectilinear area.

To pass easily, and without the encumbrance of radicals, from the first curve to the new one, we must use a technique which is always the same, and which consists of our method.

Let HE.ED be an arbitrary one of the products to be set up as ordinates on the base; since we analytically call by e the ordinate FD or its parallel HE, and by a the co-ordinate FH or its parallel DE, we will call by ea the product HE.ED. Let us equate this product ea, formed by two unknown and undetermined lines, to bu, that is to the product of the given b times an unknown u, and let us assume that u is equal to EP taken along the same line as DE: we will then have $\frac{ba}{e} = a$. But, according to the specific property of the first curve: $b^2 - a^2 = e^2$. Then, substituting for a its new value $\frac{bu}{e}$, there will result $b^2e^2 - b^2u^2 = e^4$ or, by transposition, $b^2e^2 - e^4 = b^2u^2$, the

[14]Does this sound similar to "discs and washers," by chance?

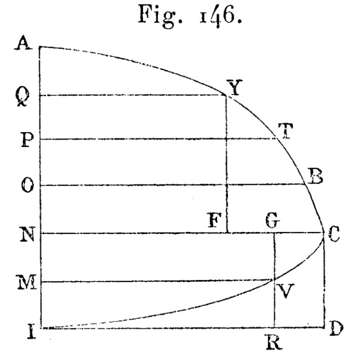
characteristic equation of the new curve HOPN, derived from the first, and for which it is proven that the sum of the values of bu set up as ordinates on b is given. Dividing by b, the sum of the values of u set up as ordinates on the base (the surface HOPN), will be given by a rectangular area, and we will thus have its quadrature.

Let us have, as the second example, $ba^2 - a^3 = e^3$ as the characteristic equation of the first curve. The sum of the values of e^3 set up as ordinates on the diameter b is given, and therefore so is the sum of the products HE^2.ED set up as ordinates on the base. But HE^2.ED is the analytic expression e^2a. Let us equate this product to b^2u, and let us assume, as above, that EP $= u$. We will have $\frac{b^2u}{e^2} = a$. If then, instead of a, we substitute its value $\frac{b^2u}{e^2}$, and if we follow the rules of the analysis, we will have $b^5u^2e^2 - e^9 = b^6e^3$, which equation forms the new curve HOPN derived from the first, and for which the sum of the products b^2u, set up as ordinates on base d, is given. Dividing by b^2, the sum of the values of u as ordinates on base d will be given, and thus the quadrature of the figure HOPN. The method is general and extends to all cases indefinitely.

But it must be noted and carefully observed that, for the transformation of curves whose ordinates on the diameter decrease in the direction of the base, analysts must follow another procedure which differs from the preceding one.

Let there be the primitive curve IVCBTYA (*fig.* 146), with diameter AI, and ordinates MV, NC, OB, PT, QY. This curve is assumed to be such that its ordinates MV on the side of the base decrease as you move towards the base, such that MV $<$ NC, and that, on the other hand, on the side of A, the curve inflects along CBYA, such that CN $>$ BO, BO $>$ PT, PT $>$ QY, etc., such that the maximum ordinate is CN.

Fig. 146.



If, in this case, we are seeking for the transformation of the squares MV^2, NC^2 into products on the base, we will no longer compare them with the product IR.RV as we did before. For the general theorem assumes that the sum $MV^2 \ldots + NC^2$ is equal to the sum of products VG.GN, since CN, the maximum ordinate, can and must be considered as the base in comparison with the curve whose summit is I. It is therefore necessary, in a curve whose ordinates decrease as you move towards the base, to compare the squares $MV^2 \ldots NC^2$ to the products GB.GN, that is to say, to arrive at this figure by an analytic equation: if we let $MI = RV = a$, $MV = RI = e$ and $CD = GR = z$ be given (this line, drawn parallel to the diameter by the extremity of the maximum ordinate, is easy to find with our methods), we will have $GV.GN = ze - ae$; consequently, the sum of squares $MV^2 \ldots NC^2$ up to the maximum ordinate will be compared to the sum of the products $ze - ae$, set up as ordinates on the base ID. The sum of the other squares CN^2, BO^2, PT^2 will be compared to the sum of the products YF.FN, by the analytic expression $ae - ze$. Having established this, we will easily derive another curve from the first curve on the [same] base; we will observe the same rule for all other powers of unknowns.

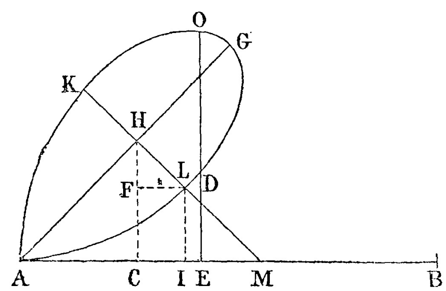
To properly demonstrate that our method provides new quadratures, which none of the moderns had the slightest inkling of, let us consider the preceding curve, whose equation is

$$\frac{b^5a - b^6}{a^3} = e^3.$$

It has been proved that the sum of the values of e^3 is given rectilinearly. By transforming them onto the base, we will have, by the preceding method, $\frac{b^2u}{e^2} = a$; after substituting the new value of a, and completing the calculations according to the rules, we will arrive at the new equation $e^3 + u^3 = beu$, which gives a curve on the side of the base.[15] This is the curve of Schooten, who has given its construction in his *Miscellanea*, section XXV, page 493 (*fig.* 147). The curved figure AKOGDCH of this author will easily be squared by the preceding rules.

[15] *donne une courbe du côté de la base*

Fig. 147.



There is also occasion to remark that, from curves where the sum of the powers of the ordinates has been found, we can deduce curves that are easy to square, not only on the base, but also on the diameter. Let us take, for example, the characteristic equation $b^2 - a^2 = e^2$ that we have already used (*fig.* 145); not only may we derive a new curve on its base having the equation $b^2e^2 - e^4 = b^2u^2$, but still another curve on its diameter, by equating the power of the ordinate e^2 to a product bu. Then the sum of the products bu, set up as ordinates upon the diameter, will be given; therefore, by dividing the sum of u set up as ordinates on the diameter by b, we will have the quadrature of the curve derived from the first curve on its diameter, whose equation will be $b^2 - a^2 = bu$. It is clear that this new curve on the diameter is a parabola. A transformation of this type not only gives new curves derived from the first, but easily leads from parabolas to hyperbolas and from hyperbolas to parabolas, as I will try to make clear.

But, just as from the curves where the sum of powers of their ordinates are given, the preceding analysis can derive curves where the sum of the simple ordinates is known — similarly, from curves whose sum of ordinates is given, we can easily arrive at curves where the sum of powers of the ordinates is known.

Let there be, for example, the curve with equation $b^2e^2 - e^4 = b^2u^2$, an equation for which, as I have established, the sum of the values of u is given. If we let $u = \frac{ae}{b}$, and if we substitute for u its new value $\frac{ae}{b}$, we will have $b^2e^2 - e^4 = a^2e^2$, and, after dividing all the terms by e^2, we will have $b^2 - e^2 = a^2$, or $b^2 - a^2 = e^2$. In this new curve, which is a circle, the sum of the values of e^2 will be given.

If from the first curve for which the sum of the ordinates is given, we desire to derive a new one where the sum of the cubes of the ordinates is known, we will still use the same method, but by using powers including unknowns.[16]

Thus, let us consider the curve that we had earlier derived from another: the curve whose equation is $b^5u^2e^2 - e^9 = b^6u^3$, for which we have established the sum of the values of u (that is, the sum of the ordinates).

To derive another curve from it where the sum of the cubes of the ordinates is known, we will let $u = \frac{e^2a}{b^2}$, and we will substitute this new value for u. By performing the calculations according to the rules of our art, we will have the equation $ba^2 - a^3 = e^3$, which will give a curve where the sum of the values of e^3 (the sum of the cubes of the ordinates) will be given.

This method not only leads to the knowledge of an infinitude of quadratures unknown among geometers until now, but it also allows the discovery of an infinite number of curves for which we can obtain the quadratures, provided that we consider as given the quadratures of simpler curves, such as the circle, the hyperbola, etc.

For example, in the equation of the circle $b^2 - a^2 = e^2$, we have, given rectilinearly, the sums of all the powers of the ordinates with powers that are even, square, biquadratic, bicubic, etc. As for the sum of the powers of odd exponents, such as e^3, e^5, it cannot be given rectilinearly, unless we assume the quadrature of the circle. It is easy to demonstrate what I have just said and to reduce it to a rule, as a corollary of the preceding method.

It also happens often enough that, in order to find the measure of a proposed curve, it is necessary to repeat the operations two or more times.

Let us take, for example, the curve determined by the following equation:

$$b^3 = a^2e + b^2e.$$

If the sum of the values of e is given, as well as the line b, then we will also know the sum of the be rectangles. By inverting the method that I presented at the beginning of this Dissertation, let us take $be = \bar{o}^2$, whence $\frac{\bar{o}^2}{b} = e$. Substituting the new value for e, there will result:

$$b^4 = a^2\bar{o}^2 + b^2\bar{o}^2.$$

Here we have a first operation, the inverse of that indicated at the beginning of the Dissertation, and which produces a new curve for which we must determine whether the sum of the values of $\bar{o}^2$ is known.

Therefore we must return to the second method, which derives the sum of the simple ordinates from the sum of the squares of the ordinates.

[16] *mais en prenant des puissances conditionnées des inconnues*

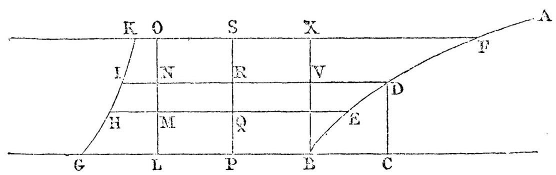
Following the preceding method put forward in the second line, let us take $\frac{bu}{\bar{o}} = a$ and let us substitute for a the new value which this method assigns to it. There will result $b^4 - b^2\bar{o}^2 = b^2u^2$, and by dividing all the terms by b^2, we will have $b^2 - \bar{o}^2 = u^2$, the equation of a circle. The sum of the values of u is thus known, if we assume the quadrature of the circle.

If we return to the first curve: $b^3 = a^2e + b^2e$, it results that the area of this curve can be squared by assuming the quadrature of the circle, and we arrive at this conclusion with ease and rapidity by using our analysis, by means of two curves different from the preceding.

The utility of everything that has preceded will be instantly recognized by a subtle analysis, both for the construction of lines equal to curves, as well as for a number of other problems which have not yet been examined in sufficient detail.

Let AB be a primary parabola (*fig.* 148), with CB as its axis, CD the ordinate equal to the axis CB and equal to parameter BV. Take BP, PL, LG equal to each other and to the axis CB, and set up on the extension of CB. Then, parallel to CD, draw the infinite lines BX, PS, LO (which will be given), and through the arbitrary point F of the curve draw the line FXSOK parallel to the axis, intersecting lines BX, PS, LO at points X, S, O. Finally, make $\frac{\mathrm{FX+XS}}{\mathrm{SO}}$ or $\frac{\mathrm{FS}}{\mathrm{SO}} = \frac{\mathrm{SO}}{\mathrm{OK}}$. Taking points D, E similarly, set $\frac{\mathrm{DR}}{\mathrm{RN}} = \frac{\mathrm{RN}}{\mathrm{NI}}$ and $\frac{\mathrm{EQ}}{\mathrm{QM}} = \frac{\mathrm{QM}}{\mathrm{MH}}$. Through points G, H, I, K,... create an infinite curve which will have the infinite line LO as its asymptote.

Fig. 148.



This curve GHIK is that whose species is defined by the preceding equation: $b^3 = a^2e + b^2e$. I therefore say, after the above-indicated re-iteration of the analytical operations, that the area of KIHGLMNO, which extends indefinitely on the side of points K, O, is equal to *twice* the circle having axis BC as its diameter. This is how I had immediately solved this question which I posed to a wise geometer.[17]

[17]Who?

I have squared the curve of Diocles by the same method, or, rather, I have reduced its quadrature to the quadrature of the circle.

But the repetition of the operations is most elegant when analysis goes from higher powers of the ordinates to lower powers, or from low to high powers. In particular, this method is applicable to find the sum of the ordinates in an arbitrarily proposed curve, as well as to many other problems of quadratures.

Let us take, for example, the curve whose equation is

$$b^2 - a^2 = e^2,$$

which you immediately see is a circle. We seek the sum of the cubes of the ordinates, that is, the sum of the values of e^3.

If the sum of the values of e^3 is given, we can, by the preceding methods, because of the nature of the power, derive from this curve another on its base, where the sum of the ordinates is given. Let us take, according to the method, $\frac{b^2\bar{o}}{e^2} = a$. Substitute the new value for a, giving $b^2e^4 - e^6 = b^4\bar{o}^2$, the equation of the curve for which the sum of the values of $\bar{o}$ would be known, under the hypothesis that the sum of the values of e^3 in the first curve were known.

Since, in this new curve, the sum of the values of $\bar{o}$ is given, we can derive a third curve from it, for which we will seek out the sum of the squares of the ordinates, and not that of the cubes of the ordinates as we had done for the first curve. Following our method for squares, we will set, as you have seen before, $\frac{eu}{b} = \bar{o}$, whence we derive $b^2e^4 - e^6 = b^2e^2u^2$. Dividing everything by e^2, we will have $b^2e^2 - e^4 = b^2u^2$, for which curve the sum of the values of e^2 is known. From this curve, let us seek another for which the sum of the ordinates is given; for example, let us take $e^2 = by$; the last equation will become $by - y^2 = u^2$. Therefore, if the sum of the values of e^2 is given, then in the current equation we will have the sum of by, and thus the sum of the values of y.

Yet in this last curve, which is clearly a circle, the sum of the values of y is given, by assuming, however, the quadrature of the circle; therefore, by returning from this last curve, where our analysis has finished, to the first, it is clear that in the circle the sum of the cubes of the ordinates is given, if we assume the quadrature of the circle. Similarly for the fifth, seventh, and the other powers of odd degree, as is easy to see. We simply increase the number of the curves, as the degree of its power increases. We will move without difficulty from the analysis to the synthesis and to the true calculation of the figure to be squared.

For the rest, it often strangely happens that we must lead the analysis through a very great number of curves in order to arrive at the simple measure for a proposed locus-equation.

For example, let $\frac{b^7a-b^8}{a^6} = e^2$.

Let us assume that the quadrature of the figure corresponding to this equation is known; the sum of the values of a is known, and therefore he sum of the values of ba, and if we let $ba = \bar{o}^2$, then we will also know the sum of values of $\bar{o}^2$. Moreover, we will have $a = \frac{\bar{o}^2}{b}$, whence we derive the equation $\frac{b^{12}\bar{o}^2-b^{14}}{o^{12}} = e^2$.

From this new curve, by the other method that we have so-often indicated, we will derive a third. The sum of the values of $\bar{o}^2$ being known, let $\frac{bu}{\bar{o}} = e$, and we will have the equation $\frac{b^{10}\bar{o}^2}{\bar{o}^{10}} = u^2$.

It is in this third curve that we will have the sum of the values of $\bar{o}$, and therefore of u. But if the sum of the values of u is given, we will have, according to the first method, the sum of the products bu. Let $bu = y^2$, whence $\frac{y^2}{b} = u$. We will have the equation$\frac{b^{12}\bar{o}^2}{o^{10}} = y^4$, the fourth curve, for which the sum of the values of y^2 will be known. By the ordinary method, let us deduce another from it. Let $\frac{bi}{y} = \bar{o}$; performing the calculations according to the rules of analysis, we will have $b^4y^4i^2 - b^4y^6 = i^{10}$, the fifth curve, for which the sum of the values of y, and thus of i, will be given.

Now, by the contrary method, already used often, let us seek another curve for which we know the sum of the squares of the ordinates; let $\frac{ia}{b} = y$ (since, lacking vowels, nothing prevents our reuse of those we have already employed), and we will have $b^2a^4 - a^6 = b^2i^4$, the sixth curve, where the sum of the values of i^2 is given.

Let us now return to the roots by the known method that we have already used several times; let $i^2 = be$; we will know the sum of the values of be, and a seventh curve $b^2a^4 - a^6 = b^4e^2$, where the sum of the values of e is known, and thus the sum of the values of a.

From this we will derive another, where the sum of the squares of the ordinates will be known.

According to the method, let $\frac{a\bar{o}}{b} = e$, whence $b^2a^4 - a^6 = b^2e^2\bar{o}^2$. Dividing all the terms by a^2, we get $b^2a^2 - a^4 = b^2\bar{o}^2$, the equation of an eighth curve for which the sum of the values of a^2 is known. With the sum of the values of a^2, we can at last derive another curve for which we know the sum of th ordinates. Let $a^2 = bu$, giving $bu - u^2 = \bar{o}^2$, the last equation, which gives a *ninth* curve, where the sum of the values of u is given. But this last curve is clearly a circle and the sum of the values of u is only known if we assume the quadrature of the circle. Therefore, by returning to the equation of the first curve, its quadrature will be known if we assume that of the last curve (the circle).

Thus we have used nine different curves to arrive at an understanding of the first.