Fermat to Pascal

September 25, 1654

Sir,

1. Do not worry that our concordance falters, you have reconfirmed it yourself by your concern that it was destroyed, and it seems to me that in responding to M. de Roberval for you, you have also responded for me.

I take the example of three gamblers, where the first needs to win one more, and the other two need two. This is the case that you object to.

I only find 17 combinations for the first and 5 for each of the two others: for, when you say that the combination *acc* is good for the first and the third players, it seems that you do not remember that everything that happens after one player has won does not matter. Yet this combination makes the first player the winner after the first round, so what does it matter that the third has won two rounds afterwards? He could win thirty more without changing things.

As you have quite well remarked, it follows that this fiction of extending the game to a certain number of rounds serves only to facilitate the rule and (according to my thoughts) to make all the chances equal, or, more intelligibly, to bring all fractions to the same denominator.

And so you have no more doubts, if instead of *three* rounds, you extend the fiction to *four*, there will be not 27, but rather 81 combinations, and you will then need to see in how many combinations the first player wins before the two others, and how often of the other players wins before his competitors. You will find that the combinations in favor of the first will total 51 and those for each of the others will be 15, which makes the same ratio as before.

If you had five rounds, or any other number that you would like, you will always find three numbers in the proportion of 17, 5, 5.

And thus I have the right to say that combination *acc* is only in favor of the first and not the third player, and that *cca* is only for the third but not for the first player, and therefore my rule of combinations is the same for three players as it is for two, and generally for any number of players.

2. You can already see by the preceding that I do not at all hesitate to make the true solution of the question of three players for which I have sent you the three decisive numbers 17, 5, 5. But because M. de Roberval will perhaps be set more at ease to see a solution without any fiction, and which can produce shortcuts in many cases, here you have an example:

The first can win after one, two, or three rounds.

If he wins in one round, then he would have to win on the first roll of a three-sided die. Such a die has three chances: this player has thus $\frac{1}{3}$ of the chance, when only one round be played.

If he plays two, then he can win in two ways, either when the second players wins the first and he the second, or when the third player wins the first and he the second. Now, two dice produce 9 outcomes: this player thus has $\frac{2}{9}$ of the odds, when two rounds be played.

If three be played, he can only win in two ways: either when the second player wins, then the third, and then him, or when the third player wins, then the second, and then him. For if either the second or the third player wins both of the first two rounds, he would win the game instead of the first player. Now, three dice make 27 differnt rolls, therefore the first player has $\frac{2}{27}$ of the odds when three rounds be played.

The sum of the chances for the first player to win is $\frac{1}{3}$, $\frac{2}{9}$ and $\frac{2}{27}$, which together make $\frac{17}{27}$.

And the rule is good and general in all cases, such that, without recourse to the fiction, the true combinations in which each number of rounds have their solution and make what I said at the commencement clear, that the extension to a certain number of rounds is nothing other than the reduction of various fractions to a same denominator. There you have the whole matter resolved in a few words, which will no doubt bring us back to friendship,[1] since we both seek only reason and truth.

3. I hope to send to you on Saint-Martin's a Summary of everything of importance that I have discovered about numbers. Allow me to be concise

[1] *bonne intelligence*

and to make myself understood to a man who understands everything in half a word.

That which you will find more important regards the proposition that any number is composed of one, two, or three triangles; of one, two, three, or four squares; of one, two, three, four, or five pentagons; of one, two, three, four, five, six hexagons, and so on to infinity.[2]

To arrive at this, it must be demonstrated that every prime number, which is one greater than a multiple of 4, is composed of two squares, for example: 5, 13, 17, 29, 37, etc.

Given a prime number of this type, such as 53, find a general rule to determine the two squares that compose it.

Every prime number, which is one greater than a multiple of 3, is composed of a square and the triple of another square, examples: 7, 13, 19, 31, 37.

Every prime number which is either 1 or 3 greater than a multiple of 8, is composed of a square and the double of another square, as for example: 11, 17, 19, 41, 43, etc.

There is no triangle in whole numbers whose area is equal to a square number.[3]

This will be followed by the invention of many propositions that Bachet confessed himself ignorant of, and which are missing in Diophantus.

I am persuaded that once you know my means of demonstrating this type of proposition, you will think it beautiful and it will give you the means to make many new discoveries; for it must be, as you know, that *multi pertranseant ut augeatur scientia*.

If I have the time, we can next speak about magic numbers, and I will recall my old thoughts on this subject.

I am with all my heart, Sir, your, etc.,

FERMAT.

The 25th of September

I hope for the health of M. de Carcavi as I do my own, and am all with him.

I am writing you from the countryside, which may slow my responses during vacation.

[2]September 1636 letter to Mersenne, **3**.

[3]Cf. Fermat's *Observations on Diophantus*.