Pascal to Fermat

August 24, 1654

SIR,

1. I cannot express to you my entire thoughts concerning the shares[1] of several gamblers by the ordinary path, and I even have some repugnance to attempting to do so, out of fear that the admirable agreement which exists between us and is so dear to me, could begin to flag, for I fear that we will be of different opinions on this subject. I would like to explain all my reasons to you, and please do me the courtesy of correcting me if I err, or agreeing if I am correct. I ask this of you completely sincerely, for I am only certain of myself when you are are on my side.

When there are only *two* players, your method, which proceeds by combinations, is very certain; but when there are *three*, I believe I have a demonstration that it is not correct, although perhaps only because you proceed in a manner that I do not understand. But the method that I have shown you and which I always use is common to all imaginable conditions of all sorts of partitions, while the method of combinations (which I use only in particular situations where it is shorter than the general theory) works only in this case, and in no others.

I am sure of what I will put forward, but there is needed on my part a bit of explanation, and on your part, a bit of patience.

2. Here is how you proceed when there are *two* players:

If two players, playing in several rounds, find themselves in the state that the first needs *two* to win, and the second needs *three,* you say that to find

[1] I use "shares," "partitions," and "allocations" equally here. The question is of splitting the wagers that gamblers have made, if the game be called off before it has ended according to its usual rules.

the shares, it must be determined in how many rounds the game will be absolutely decided.

It is easily worked out that it will be in *four* rounds, whence you conclude that it is necessary to see how four rounds may be combined among two players and see in how many combinations the first player wins and in how many the second player wins, and then divide the money in this proportion. I would have had difficulty in trying to understand this thought process, had I not already known it myself; also you have written it. Therefore, to see how many ways four rounds may be combined among two gamblers, it is necessary to imagine that they play with a two-faced die (since there are only two players), as in heads and tails, but I don't know what that means., and that they roll four of these dice (since they are playing four rounds); and now it is necessary to see how many different combinations of these dice there may be. This is easy to work out: they have *sixteen*, which is the second degree of *four*, which is its square. For let is say that one of the faces is marked a, in favor of the first player, and the other b, for the second player; therefore these four dice can land in these sixteen configurations:

a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b
a	a	a	a	b	b	b	b	a	a	a	a	b	b	b	b
a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
a	b	a	b	a	b	a	b	a	b	a	b	a	b	a	b
1	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2

and, because the first player must win two rounds, all the faces which have two "*a*"s make him the winner: therefore he has 11 for himself; and because three are needed for the second player, all those with three "*b*"s allow him to win: therefore he has 5. Therefore the sum must be divided in the ratio of 11 to 5.

There is your method when there are *two* players; and you say that when there are more, it will not be difficult to decide their shares by the same method.

3. On this point, Sir, I must tell you that this allocation for two players, based on combinations, is very correct and very good; but if there be more than two players, it will not always be correct, and I will tell you the reason for this difference.

I have communicated your method to our gentlemen. M. de Roberval raised this objection to it:

That it is wrong to make the division based on the supposition that there will be *four* rounds played, considering that when there are *two* lacking for one player and *three* for the other, it is not necessary to play *four* rounds, since the game may end in only *two* or *three* rounds, or, in truth, perhaps *four;*

And since he does not see why it is claimed that the division is made correctly on the pretended condition that *four* rounds will be played, considering that the natural condition of the game is that the game does not continue after one of the players has won, and that if it is not false, the method is at least not demonstrated, in that he has some suspicious that we have made a paralogism.

I responded to him that I did not base myself on this method of combinations, which is truly not in its proper place on this occasion, but rather on my other universal method, from which nothing escapes and which brings its demonstration with itself, which finds exactly the same division as the method of combinations; and beyond that I have demonstrated to him the truth of the division between two players by combinations in this manner:

Is it not true that if there are two players, finding themselves in the hypothesized condition where the first needs to win *two* and the second *three,* and they mutually agree to play *four* complete rounds, which is that they will roll four dice with two faces all at once, is it not true, I say, that if they had decided to play four rounds, the partition must be, as we have said, according to the number of configurations favorable to each?

He found himself to be in agreement and this is indeed demonstrative; but he denied that the same thing remained true without the obligation of playing *four* rounds. I therefore told him:

Is it not clear that the same players, not being compelled to play four rounds, but desiring to end the game as soon as one has won, could without damage or advantage decide to play the entire four rounds and that this would in no way change their condition? For, if the first wins the first two rounds out of the *four* and has thus won, would he refuse to play two more rounds, considering that if he wins them, he wins no more, and if he loses, he wins no less? These two that the other wins will not suffice, since he needs to win three, and thus in four rounds they cannot both win the number that they need.

Certainly it is easy to consider that it is absolutely equal for both players to play the game naturally, which is to end as soon as one has attained the required number, or to play four entire rounds: therefore, since these two conditions are equal, the partition must be the same in both cases. Now, it is fair when they are obliged to play four rounds, as I have shown: therefore it is also fair in the other case.

There you have my demonstration, and notice that this demonstration is based upon the equality of the two conditions, actual and imagined, as far as regards the two players, and in both the same will win, and if one wins or loses in one, he will win or lose in the other, and never will both succeed.

4. Let us take the same approach for *three* players, and let us say that the first player needs *one* point, the second *two*, and the third *two* as well. To make the partition, according to the method of combinations, it is first necessary to find in how many rounds the game will be decided, as we had done when there were two players: this will be *three* rounds, for they would be unable to play *three* rounds without a decision necessarily being arrived at.

Now it must be seen how three rounds may be combined among three players and how many are favorable for the one, for the other, and for the last, and according to this proposition, to distribute the money as we did in the hypothesis with two players.

It is easy to see how many combinations there are: it is the third power of 3, that is, 27. For, if you throw *three* dice at once (because three rounds must be played), which each have *three* faces (since there are three players), one marked a for the first player, b for the second, c for the third, it is manifest that these three dice rolled together can land in 27 different positions, to wit:

a	a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b	b	c	c	c	c	c	c	c	c	c
a	a	a	b	b	b	c	c	c	a	a	a	b	b	b	c	c	c	a	a	a	b	b	b	c	c	c
a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
1	1	1	1	1	1	1	1	1	1	1	1	1			1			1	1	1	1			1		
				2						2		2	2	2		2						2				
								3									3			3			3	3	3	3

Now, the first needs only *one* round: therefore all the positions where there is an a are for him: therefore he has 19.

The second needs *two* rounds: therefore all the rolls with two b are for him: he then has 7.

The third needs *two* rounds: therefore all the rolls with two c are for him: he then has 7.

If from this we conclude that the money must be divided according to the proportion of 19, 7, 7, we will err most uncouthly, and I would not even consider thinking that you would do it this way; for there are some faces favorable for both the first and the second players, such as *abb*, for the first has the *a* he requires, and the second has two *b* which he needs, and such is *acc* for the first and the third.

Therefore these rolls which are common to two players cannot be worth the entire sum to both, but only a half. For, if the roll *acc* is made, the first and the third will have the same right to the sum, each having won enough rounds, and therefore they would divide the money in half; but if the roll *aab* is made, the first wins all. It is therefore necessary to calculate thus:

There are 13 rolls which give the entirety to the first and 6 which give him half, and 8 which give him nothing. Therefore if the entire sum is a pistole, there are 13 faces which give him one pistole, 6 which each give $\frac{1}{2}$ pistole, and 8 which are worth nothing.

Therefore, in the case of partition, we must multiply:

	13 by one pistole,	which makes	13
	6 by one half,	which makes	3
	8 by zero,	which makes	0
	—		—
Sum:	27	Sum...	16

and divide the sum of the values, 16, by the number of rolls, 27, which makes the fraction $\frac{16}{27}$, which is the amount belonging to the first in the case of a partition, namely 16 pistoles out of 27.

The part of the second and third players is found to be the same:

There are	4 rolls which are worth 1 pistole :	multiply...	4
There are	3 rolls which are worth $\frac{1}{2}$ pistole :	multiply...	$1\frac{1}{2}$
There are	20 rolls which are worth nothing	....	0
	—		—
Sum...	27	Sum...	$5\frac{1}{2}$

Therefore the second player deserves 5 pistoles and $\frac{1}{2}$ out of 27, and as many for the third, and these three sums, $5\frac{1}{2}$, $5\frac{1}{2}$ and 16, combined, make up 27.

5. There you have, I believe, the correct manner of making partitions by combinations according to your method, if you do not have something else

on this subject that I could not determine. But, if I do not mistake myself, this partition is unfair.

The reason is that a false assumption was made, which is that there are necessarily *three* rounds to play, although the natural way of playing is that play ends when one of the players has attained the required number of rounds.

It is not that it will never happen that three rounds be played, but it may occur that only one or two rounds be played.

But why is it, one might ask, that we may not make the same imaginary assumption as before when there were two players. Here lies the reason:

In the actual case of three players, there is only one who may win, for the condition is that once a player has won, the game ceases. But, in the false condition, two players may win the required number of rounds: namely, if the first player wins the one that he needs and one of the others wins the two that he needs; for they will have played three rounds, whereas when there are only two players, the false and the true conditions confer the same advantages on the players in every respect; and this is what makes the extreme difference between the false and the true conditions.

If the players are in the hypothesized condition, i.e., the first needs *one* and the second and third need *two*, and if they come to a mutual agreement that they will play *three* entire rounds, and that he who reaches the needed number alone will win the entire sum, while if two do so, they will split it equally, *in this case,* the partition should be done as I have given it, that the first gets 16, the second $5\frac{1}{2}$, and the third $5\frac{1}{2}$ of the 27 pistoles, and the demonstration is given if this condition be assumed.

But if they play with the condition not of necessarily playing three rounds, but just until one has attained the needed number, and then stop playing without giving another player the means of also succeeding, then to the first will belong 17, 5 to the second, and 5 to the third, of the 27.

And this is found by my general method which also determines that in the preceding condition, the first deserves 16, $5\frac{1}{2}$ for the second, and $5\frac{1}{2}$ for the third, without using combinations, for it [the general method] works by itself without any problem.

6. There you have, Sir, my thoughts on this subject on which I have no advantage over you save that I have thought much more upon it; but this is little in your respect, since your first views are more penetrating than the prolixity of my efforts.

I do not allow myself to open my thoughts in order to await your judgment. I believe to have let you know that the method of combinations works for two players, by accident, as also sometimes among three players, as when the first needs *one*, the second *one*, and the third *two*, because in this case the number of rounds to complete the game does not allow two to win; but it is not more general and does not work generally, but only in those cases in which there is an obligation of playing exactly a certain number of rounds.

So, since you did not have my method when you proposed to me the shares of several players, but had only the method of combinations, I fear that we will be of different opinions on this subject.

I urge you to send me whatever progress you make in your studies. I will receive your response with respect and with joy, even if your opinion contradicts my own. I am, etc.