Observations on Diophantus

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1. – Porisms of Bachet, Book III, Definition 6

A right triangle in numbers (that is the set of three rational numbers a, b, c, connected by the relation: $a^2 = b^2 + c^2$) is said to be formed by two numbers p and q, if:

$$a = p^2 + q^2,\ b = p^2 - q^2,\ c = 2pq.$$

We can form a triangle with three numbers in arithmetic progression, by composing it, according to definition 6, with the middle term and the difference of the two terms; for the product of the three terms times the difference will be equal to the area of said triangle, and, accordingly, if the difference is unity, the area of the triangle will be represented by the product of the three terms.

2. – Diophantus, II, 8.

Solve by rational numbers the indeterminate equation: $x^2 + y^2 = a^2$.

On the contrary, it is impossible to divide either a cube into two cubes, a biquadratic into two biquadratics, or, generally, any power superior to the square into two powers of the same degree; I have discovered a truly marvelous demonstration of this which this margin is too narrow to contain.[1]

3. – Diophantus, II, 10

Solve: $x^2 + y^2 = a^2 + b^2$.

[1] Yes, this is his statement of what has come to be known as "Fermat's Last Theorem."

Can a number which is the sum of two cubes also be divided into two other cubes? There is a difficult problem whose solution has certainly been ignored by Bachet and Viète, and perhaps by Diophantus himself; I have solved it further on in my Notes on problem IV, 2.

4. – Diophantus, III, 10

Solve: $x+y+a=\square, y+z+a=\square, z+x+a=\square, x+y+z+a=\square$.

I have indicated, in my Note on problem V, 30, how one can find four numbers such that the sum of any two among them, increased by a given number, makes a square.

5. – Diophantus, III, 11

Solve the preceding problem, supposing a to be negative.

My Note on V, 31 shows how one can find four numbers such that the sum of any two among them, decreased by a given number, makes a square.

6. – Diophantus, III, 17

Solve $xy+x+y=\square, yz+y+z=\square, zx+z+x=\square$.

In Diophantus there is another problem, V, 5, on the same subject[2]. However we do not know if he omitted, while knowing it, the following problem, or, more probably, if he had given the solution in one of his thirteen Books:

> Find three squares such that the product of any two among them, increased by the sum of the same two squared, makes a square.

I can give an infinite number of solutions to this problem. Here is one, for example: the three squares $\frac{3\,504\,384}{203\,401}$, $\frac{2\,019\,241}{203\,401}$, 4, satisfy the proposed question.

One could indeed go further and extend the question of Diophantus. Thus I have treated the following question generally and I can provide an infinite number of solutions to it:

[2] V, 5, Diophantus supposes that the unknowns of III, 17 are squares; he further adds the conditions: $xy+z=\square, yz+x=\square, zx+y=\square$. — (Note from the editors of the Œuvres)

> Find four numbers such that the product of any two among them, increased by the sum of the same two numbers, makes a square.

Following V, 5, we will look for three squares such that the product of any two among them, augmented by the sum of the same two squared, makes a square. Let there be for example the three squares given by Diophantus: $\frac{25}{9}$, $\frac{64}{9}$, $\frac{196}{9}$; we will take them for the first three numbers of our problem; let x be the fourth; by forming its product with each of the preceding and adding the sum of the two factors, we will have

$$\tfrac{34}{9}x + \tfrac{25}{9} = \square,\ \tfrac{73}{9}x + \tfrac{64}{9} = \square,\ \tfrac{205}{9}x + \tfrac{196}{9} = \square,$$

a triple equation, whose treatment I have taught in my Note on VI, 24.

7. – Commentary of Bachet on Diophantus, III, 22.

Any prime number, of the form $4n+1$, is the hypotenuse of a right triangle in one way; its square is such a hypotenuse in two ways, its cube in three ways, its biquadratic in four ways, and so on indefinitely.[3]

The same prime number and its square are, in one way, the sum of two squares; its cube and biquadratic are the sum of two squares in two ways; its fifth and sixth power are so in three ways, and so on indefinitely.

If a prime number, which is the sum of two squares, is multiplied by another prime number, which is also the sum of two squares, their product will be, in two different ways, the sum of two squares; if the multiplier is the square of the second prime number, the product will be the sum of two squares in three different ways; if the multiplier is the cube of the second prime number, the product will be the sum of two squares in four different ways, and so on and so forth.

Following this, it is easy to demonstrate *in how many different ways a number may be the hypotenuse of a right triangle.*

We will take all the prime divisors of this number which are of the form $4n + 1$: 5, 13, and 17, for example.

If the given number is divided by powers of these prime divisors, it is then necessary to take these powers in the place of the simple factor; for example, let us suppose that the given number be divided by the cube of 5, by the square of 13, and by 17 simply.

[3] For example, 5 is the hypotenuse of the 3,4,5 triangle, while 25 is the hypotenuse of the 15,20,25 triangle as well as the 7,24,25 triangle.

We will take the exponents of all the factors, namely: for 5, the exponent 3 of the cube, for 13, the exponent 2 of the square, and for 17, simply unity.

We will then order said exponents as we please; let it be, for example, the order 3, 2, 1.

We will multiply the first by the second, then double it, and add the sum of the first and the second: giving us 17. We will multiply 17 by the third, double and add the sum of 17 and the third: giving us 52. The given number will be the hypotenuse of 52 different right triangles. The process will be the same whatever may be the number of factors and whatever may be their powers.

The other prime numbers, which are not of the form $4n + 1$, as well as their powers, do not at all increase or decrease the number we are trying to find.

Find a prime number which is the hypotenuse in a given number of ways.

Let us find a number which is the hypotenuse in seven different fashions.

I double the given number 7, giving me 14. I add 1, which makes 15. I take all the prime divisors of 15, which are 3 and 5. I take unity from each of them, and I take the half of what remains: I have 1 and 2. I now take as many prime factors I have here distinct numbers, namely two, and I multiply these factors while raising them to the exponents 1 and 2; provided that these factors be of the form $4n+1$, I will thus have (by multiplying one by the square of the other) a number that satisfies the proposed question.

From this it is easy to find the minimum number that is the hypotenuse in as many ways as you like.

Find a number which is the sum of two squares in a desired number of ways.

Let 10 different ways be proposed; I take all the prime factors of its double, 20, giving me $2 \cdot 5 \cdot 5$. I subtract unity from each of these numbers, giving $1 \cdot 1 \cdot 4$. Consequently I will have to take three prime numbers of the form $4n+1$, 5, 13, 17 for example; because of the exponent 4, I will take the fourth power of one of these numbers, I will multiply it by the other two, and I will thus have the sought-for number.

After this, it is easy to find the smallest number which is the sum of two squares in a desired number of ways.

To know *in how many different ways a given number is the sum of two squares,* here is the method.

Let the proposed number be 325. Its prime divisors of the form $4n + 1$ are: 5 squared and 13 simply. I take the exponents: 2, 1. I add their product to their sum, which makes 5; I add unity, which makes 6; I take the half, 3. The given number will be the sum of two squares in *three* different ways.

If we have three exponents, for example: 2,2,1, here is how to calculate. I take the product of the first two and add their sum, which makes 8. I multiply 8 by the third and add the sum of the factors, which makes 17. Finally, I add unity, which make 18, whose half is 9. The proposed number will be the sum of two squares in *nine* different ways.

If the last number whose half must be taken turns out to be odd, then remove unity and take the half of what remains.

The following problem may also be proposed: *Find a whole number whose sum with another given number makes a square, and which is also the hypotenuse of as many right triangles as desired.*

The question is difficult. If, for example, it is demanded to find a number which is the hypotenuse in two ways, and which, augmented by 2, makes a square, then 2023 is the number satisfying these conditions, and there are an infinite number of others, such as 3326, etc.

8. – Commentary of Bachet on Diophantus, IV, 2

> 1. To solve: $x^3 + y^3 = a^3 - b^3$, make $x = \frac{3a^3b}{a^3+b^3} - b$, $y = a - \frac{3ab^3}{a^3+b^3}$. In order that the two numbers x, y be positive, it must be that $a^3 > 2b^3$.

By repeating the operation, it is easy to free oneself from the condition, and to solve this question generally as well as the following, which neither Bachet nor Viète himself were able to do.

Let the two cubes 64 and 125 be given; we demand two others whose sum is equal to the difference between the two given cubes.

Following the process given by Bachet for his problem 3 on the following page, we will seek two other cubes whose difference is equal to that between the two given cubes. Bachet gave these two cubes: $\frac{15\,252\,992}{250\,047}$ and $\frac{125}{250\,047}$. By construction, their difference is equal to the difference between the two given cubes; but, after having found them by the operation indicated for problem 3, when double the smaller is not greater than the larger, they can be used as the givens in problem 4.

We will thus have two given cubes, and we will look for two others whose sum is equal to the difference between the givens; the condition indicated for problem 1 being satisfied, the solution will be found without difficulty. But the difference between the two cubes found by problem 3 is equal to the difference between the two originally given cubes 64 and 125. Thus nothing prevents our constructing two cubes whose sum is equal to the difference between the given 64 and 125, which would without doubt astonish Bachet himself.

What's more, by passing through the three problems circularly and repeating the operations indefinitely, an infinite number of pairs of cubes satisfying the same condition will be formed. Indeed, after having found in the last place our two cubes whose sum is equal to the difference of the givens, we can (problem 2) look for two others whose difference is equal to the sum of our two cubes, that is to say equal to the difference of those originally given; from the difference we will go again to the sum, and so on indefinitely.

9. – Same commentary

2. To solve: $x^3 - y^3 = a^3 + b^3$, make $x = \frac{3ab^3}{a^3-b^3} + a$, $y = \frac{3a^3b}{a^3-b^3} - b$.
3. To solve: $x^3 - y^3 = a^3 - b^3$, make $x = \frac{3a^3b}{a^3+b^3} - b$, $y = \frac{3ab^3}{a^3+b^3} - a$.

In order that x, y be positive, it is necessary that $a^3 < 2b^3$.

The condition imposed by the solution of this problem 3, is not legitimate, as I will show by calculating as for problem 4.

What's more, after what has preceded, I will happily solve the following problem, whose solution was ignored by Bachet:

Divide a number which is the sum of two cubes, into two other cubes, and that in an infinite number of ways, by continually repeating the operations as I have indicated above.

Thus let it be sought to find two cubes whose sum is equal to the sum of the two cubes 8 and 1. First, (for problem 2), I will seek out two cubes whose difference is equal to the sum of the givens; I will find $\frac{8000}{343}$ and $\frac{4913}{343}$. Since the double of the smaller exceeds the greater, we are brought to problem 3, whence we will go to problem 4, and thence we will have the solution.

If we would like a second solution, we will go again to problem 2 and so on as follows.

To show that the condition posed by problem 3 is not legitimate, let us find, given the two cubes 8 and 1, two other cubes whose difference is equal to that of the givens.

Bachet would without doubt say that the problem is impossible. With my method, I have nonetheless found the following two whose difference is $7 = 8 - 1$. The two cubes are $\frac{2\,024\,284\,625}{6128\,487}$ and $\frac{1\,981\,385\,216}{6128\,487}$, and their roots are $\frac{1\,265}{183}$ and $\frac{1\,256}{183}$.

10. – Commentary of Bachet on Diophantus, IV, 11

BACHET: Solve $\frac{x^3+y^3}{x+y} = a$, supposing a to be of the form p^2 or $3p^2$.

This condition must be completed in the manner I have indicated further on for that of the following problem.[4] It is not shocking that Bachet did not perceive the general method, which is really difficult; but he should have at least warned the reader that his solution is only particular.

11. – Diophantus, IV, 12

Solve: $x^3 - y^3 = x - y$.

If one seeks *two biquadratics whose difference is equal to the difference between their roots,* this question can be solved by employing the technique of my method.

Let us seek, indeed, two biquadratics whose difference is a cube, and such that the difference between their roots is 1. We will find, by the first operation, the roots $-\frac{9}{22}$ and $\frac{13}{22}$. Since the first of these two numbers is affected by the sign $-$, we will repeat the operation following my method, by equating the first root to $x - \frac{9}{22}$, the second to $x + \frac{13}{22}$, and we will thus obtain the positive numbers satisfying the problem.

12. – Commentary of Bachet on Diophantus, IV, 12

BACHET: Solve $\frac{x^3-y^3}{x-y} = a$, supposing a to be of the form p^2 or $3p^2$.

The condition is not legitimate, because it is not general. We must add "or that the number expressing the ratio be a multiple of a square by a prime number of the form $3n+1$ (such as 7, 13, 19, 37, etc.), or by a product of prime numbers of this form (as are products 21, 91, etc.)." The demonstration and the solution of the problem depend on my method.

13. – Diophantus, IV, 17

Solve: $x_1 + x_2 + x_3 = \square$, $x_1^2 + x_3 = \square$, $x_2^2 + x_3 = \square$, $x_3^2 + x_1 = \square$.

[4] Observation 12

The problem may, perhaps, be solved more elegantly as follows:

Let us take $x_1 = x$, $x_2 = 2x + 1$, such that $x_1^2 + x_2 = \square$. For x_3, let us choose the coefficient of x and the constant term arbitrarily, such that $x_2^2 + x_3 = \square$. For example let $x_3 = 4x + 3$.

We have thus satisfied two conditions; furthermore, it is necessary that we have

$$x_1 + x_2 + x_3 = \square \qquad \text{and} \qquad x_3^2 + x_1 = \square.$$

But

$$x_1 + x_2 + x_3 = 7x + 4, \qquad x_3^2 + x_1 = 16x^2 + 25x + 9.$$

We therefore have a double equation where the constant terms are squares, whose solution is consequently simple, by bringing these terms to be equal to the same square.

By the same process, we can extend the problem to 4 numbers and indeed to as many as we like. It suffices to do so such that the sum of the independent terms of x, in the expressions of the different numbers, makes a square; which is very easy.

14. – Diophantus, IV, 18

Solve: $x_1 + x_2 + x_3 = \square$, $x_1^2 - x_2 = \square$, $x_2^2 - x_3 = \square$, $x_3^2 - x_1 = \square$.

The mode of reasoning that I employed for the preceding question equally permits solving this one as well, and permits extending it to as many numbers as you like.

15. – Diophantus, IV, 20

Solve: $x_1x_2 + 1 = \square$, $x_2x_3 + 1 = \square$, $x_3x_1 + 1 = \square$.

Let it be proposed to find three numbers such that the product of any two among them, increased by unity, makes a square, and that, furthermore, each of these three numbers themselves, increased by unity, makes a square.

I will add a solution of this question, which has already been treated.

Let there be an indeterminate solution of the present problem of Diophantus, chosen such that x_1 and x_3, the independent terms of x, each increased by unity, make squares. Let there be, for example, the three indeterminate numbers:

$$x_1 = \frac{169}{5184}x + \frac{13}{36},\ x_2 = x,\ x_3 = \frac{7225}{5184}x + \frac{85}{36}.$$

It is clear that they furnish a solution to this problem IV, 20. Furthermore, it is now necessary to satisfy the following conditions

$$x_1 + 1 = \Box,\ x_2 + 1 = \Box,\ x_3 + 1 = \Box,$$

that is to say, a triple equation, which will be easy to solve by my method, since the independent term of x, after the addition of unity, is square in each of the expressions.

16. – Diophantus, IV, 21

Solve: $x_1x_2 + 1 = \Box$, $x_1x_3 + 1 = \Box$, $x_1x_4 + 1 = \Box$, $x_2x_3 + 1 = \Box$, $x_2x_4 + 1 = \Box$, $x_3x_4 + 1 = \Box$.

First, let us seek three numbers such that the product of two among them, augmented by unity, makes a square. For example, let us take the numbers 3, 1, 8.

Now let us seek a fourth number such that its product with each of the three numbers already found makes a square after the addition of unity. Let x be this number; we will have

$$3x + 1 = \Box,\ x + 1 = \Box,\ 8x + 1 = \Box,$$

a triple equation whose solution can be obtained by the method I invented. See my Note on problem VI, 24.

17. – Diophantus, IV, 23

Solve: $x_1x_2x_3 + x_1 = \Box$, $x_1x_2x_3 + x_2 = \Box$, $x_1x_2x_3 + x_3 = \Box$.

This problem can be solved not only without the lemma of Diophantus, but even without a double equation.

If we have

$$x_1x_2x_3 = x^2 - 2x,\ x_1 = 1,\ x_2 = 2x,$$

we will satisfy two of the conditions of the problem.

To obtain x_3, we must now divide $x_1x_2x_3$, that is $x^2 - 2x$, by x_1x_2, that is $2x$. It follows that $x_3 = \frac{1}{2}x - 1$, and, by adding $x_1x_2x_3$, we will have

$$x^2 - \tfrac{3}{2}x - 1 = \square.$$

From this it is necessary that the value of x exceed 2, by reason of the positions already made. We will therefore form the root of the square $\square$, by removing from x an arbitrarily chosen number which is greater than 2. The rest is evident.

18. – Commentary of Bachet on IV, 31

> Bachet (empirical proposition): Every number is either a square, or the sum of 2, 3, or 4 squares of whole numbers.

What's more, there is a very beautiful and altogether general question which I have been the first to discover.

Every number is: either triangular, or the sum of 2 or 3 triangles;

Either square, or the sum of 2, 3, or 4 squares;

Either pentagonal, or the sum of 2, 3, 4, or 5 pentagons; and so on indefinitely, whether it be of hexagons, heptagons, or any polygons; this marvelous proposition can be enunciated generally by means of the number of angles.

I cannot here give the demonstration, which depends on numerous and abstruse mysteries of he Science of Numbers. I have the intention of dedicating an entire Book to this subject and thus, in this part of Arithmetic, I intend to make shocking progress beyond the formerly known limits.

19. – Diophantus, IV, 35

> Solve: $x_1 + x_2 + x_3 = 6$, $x_1x_2 + x_3 = \square$, $x_1x_2 - x_3 = \square$.

We can calculate more easily as follows: Arbitrarily divide the given number 6 into two numbers, for example, into parts 5 and 1. Divide the product of these parts diminished by unity, that is 4, by the given number 6, giving $\frac{2}{3}$. Subtract this quotient from both 5 and 1. The two remainders $\frac{13}{3}$ and $\frac{1}{3}$ can be taken for the first two parts of the number to divide. The third will then be $\frac{4}{3}$.

20. – Commentary of Bachet on Diophantus, IV, 44

Solve: $(x_1+x_2+x_3)x_1 = \frac{\alpha(\alpha+1)}{2}$; $(x_1+x_2+x_3)x_3 = \beta^2$; $(x_1+x_2+x_3)x_3 = \gamma^3$, with the condition that α be whole, x_1, x_2, x_3, β, γ can simply be rational.

Taking $x_1+x_2+x_3 = x^2$ and $\beta = x^2 - z^2$, we will arrive at the condition

$$\frac{\alpha(\alpha+1)}{2} = 2z^2x^2 - \gamma^3 - z^4;\text{ whence } (2\alpha+1)^2 = 16z^2x^2 - 8\gamma^3 - 8z^4 + 1.$$

We will reach a solution by equating this last expression to $(4zx-\delta)^2$; but α can never be obtained as a whole number except by taking $\delta = 1$.

Bachet has not made enough sufficiently rigorous trials. Indeed let us take for γ^3 an arbitrary cube whose root is of the form $3n+1$.

We will have, for example, to equate $2x^2 - 344$ to a triangle $[\frac{\alpha(\alpha+1)}{2}]$ and $16x^2 - 2751$ to a square $[(2\alpha+1)^2]$. Now one can, if desired, take as the root of this square, $4x-3$, etc.

Indeed nothing prevents generalizing the method and taking in place of 3 another odd number completely arbitrarily, remembering to choose the consequent cube.

21. – Commentary of Bachet on Diophantus, IV, 45

Diophantus teaches in this problem to treat the *double equation*

$$ax + b = \square,\ a_1x + b_1 = \square,$$

for the case where a and a_1 are different and where furthermore the ratio of a to a_1 is not a square, but by assuming that b and b_1 are unequal squares. Bachet shows that the solution is equally possible, b and b_1 being arbitrary: first if, by supposing $a > a_1$, the ratio of $ab_1 - ba_1$ to $a - a_1$ is square; secondly if, with the same hypothesis that $a > a_1$, the ratio of $a_1b - b_1a$ to a_1 is square.

But let us propose, for example, the double equation

$$2x + 5 = \square,\ 6x + 3 = \square;$$

we can take the squares $16 = 2x + 5$, $36 = 6x + 3$; and there are an infinite number which similarly satisfy the question. Moreover, it is not difficult to give a general rule for problems of this type such that the conditions posed by Bachet are scarcely worthy of him, for one could easily extend to an infinite number of cases, even to all possible cases, that which he found for only two cases.

22. – Diophantus, V, 3

Solve $x_1x_2 + a = \square$, $x_2x_3 + a = \square$, $x_3x_1 + a = \square$, $x_1 + a = \square$, $x_2 + a = \square$, $x_3 + a = \square$.

From this solution, it is easy to deduce the solution of the following question:

Find four numbers such that the product of any two among them, increased by a given number, makes a square.

Indeed, let there be taken for three of these numbers, those which will have been found for the problem of Diophantus and which will then also satisfy the condition that each of them, augmented by a given number, makes a square. Let $x + 1$ be the fourth number to look for; we will have a triple equation that is easy to solve by my method. See the Note on problem VI, 24.

Thus, we will have a solution for the question proposed by Bachet on III, 12, and since the process is also more general, it has the superiority over that of Bachet that the three first numbers, each increased by a given number, make squares.

All the same, I do not yet know if the problem can be solved by posing the condition that the fourth number, increased by the given, also makes a square; this is a study that remains to be made.

23. – Diophantus, V, 8

Construct three numerical right triangles with equal areas.

But can we find four such triangles, or even a greater, or perhaps an infinite number of them? Nothing appears to prevent this from being possible; therefore it is to be examined more deeply.

I have solved the problem; what's more, for an arbitrary given triangle, I can provide an infinite number of triangles of the same area. Let the area be 6, for example, in the 3, 4, 5 triangle. Here is another of the same area: $\frac{7}{10}$, $\frac{120}{7}$, $\frac{1201}{70}$, or if you prefer the same denominator: $\frac{49}{70}$, $\frac{1200}{70}\frac{1201}{70}$.

Here is the procedure which, without exceptions, can be applied indefinitely. Let there be an arbitrary triangle, with hypotenuse z, base b, and height d. We will derive another non-similar triangle from it, having the same area, by making this new triangle with the numbers z^2 and $2bd$, except dividing the fourth-degree expressions which represent the sides by $2zb^2 - 2zd^2$. The triangle thus obtained will always have an area equal to that of the triangle from which it is derived.

From the second triangle determined in this way, we will derive, by the same method, a third triangle; from this third a fourth; from the fourth a fifth, and we will thus have an infinite series of non-similar triangles with the same area.

In order that no one may doubt that it is possible to give more than three, for those of Diophantus: 40,42,58, 24,70,74, 15,112,113, I will add a fourth of the same area: hypotenuse $\frac{1412881}{1189}$, base $\frac{1412880}{1189}$, height $\frac{1681}{1189}$.

If we reduce all these numbers to the same denominator, we will have, in whole numbers the following four triangles of the same area:

1st	47 560	49 938	68 962
2nd	28 536	83 230	87 986
3rd	17 835	133 168	134 357
4th	1 681	1 412 880	1 412 881

We can find an infinite number of them of the same area by continuing the application of this procedure, and from there we can extend the following problem of Diophantus beyond the boundaries that he set for it.

Here, obtained by another procedure, is a triangle whose area is six times a square, as triangle 3, 4, 5:

$$2\,896\,804,\ 7\,216\,803,\ 7\,776\,485$$

24. – Diophantus, V, 9

> Find three numbers such that the square of each of them, either increased or decreased by the sum of the three numbers, makes a square.

After what I have said above, it is clear that I can solve the problem:

Find as many numbers as desired, such that the square of each of them, either increased or decreased by the sum of all these numbers, makes a square.

Bachet probably did not know the solution of this problem; otherwise he would have generalized the question of Diophantus, as he had done for IV, 31, and others.

25. – Commentary of Bachet on Diophantus, V, 12

> Doubts on the question of knowing whether a number which, like 21, is neither square, nor the sum of two whole squares, can be divided into two squares.

The number 21 cannot be divided into two fractional squares. I can demonstrate it very easily. More generally, no number divisible by 3 but not by 9, can be the sum of two squares, whether they be whole-number squares or fractional squares.

26. – Same Commentary

> On the conditions imposed on the choice of the given number a for the possibility of the problem: $x + y = 1,\ a + x = \Box,\ a + y = \Box$.

Here is the true condition, that is to say, the condition which is general and which excludes all the numbers which cannot be chosen:

The given number must not be odd, and the sum of its double and unity, after division by the greatest square which enters in as a factor, must not be divisible by a prime number smaller by unity than a multiple of 4.

27. – Commentary of Bachet on Diophantus, V, 14

> On the conditions imposed on the choice of the given number a for the possibility of the problem: $x+y+z = 1, a+x = \Box, a+y = \Box, a+z = \Box$.

The condition posed by Bachet is not, itself, sufficient: what's more, he did not make his trials with enough care, for his rule does not exclude the number 37, which, however, cannot be taken.

Here is how one must conceive the true condition:

Let us take two geometric progressions with ratio 4, and whose first terms are 1 and 8: then let us superpose the terms as follows:

1,	4,	16,	64,	256,	1024,	4096,	etc.,
8,	32,	128,	512,	2048,	8192,	32768,	etc.

First, I consider the first term of the second progression, 8; the given number must neither be the double of 1 (the term set above 8), nor equal to the sum of a multiple of 8 and twice 1.

In the second place, I consider the second term of the second progression, 32, and I take the double of the term 4 set above it; I add to this double, 8, the sum of the terms which precede in the same, higher, progression (in this case, this sum reduces to unity); thus I have 9.

Taking then the numbers 32 and 9, I say that the given number must be neither 9, nor the sum of 9 and a multiple of 32.

Now I consider the third term of the second progression, 128; I take the double, 32, of the number 16 which is set above it; I add the sum of the preceding terms in the same upper progression, that is, 1 and 4, giving me 37. Taking then the two numbers 128 and 37, I say that the given number must neither be 37, nor the sum of 37 with a multiple of 128.

Again, I consider the fourth term of the second progression; the same procedure will give me the numbers 512 and 149. It will then be necessary that the given number neither be 149, nor the sum of 149 with a multiple of 512.

There you see the uniform method whose application must always be followed, and which was neither indicated by Diophantus in its generality, nor understood by Bachet himself; the trials of the latter were even faulty, not only for the number 37, as I have already remarked, but also for 149 and the others, which fall equally within the limits of the trials that he claims to have made [up to 325].

28. – Diophantus, V, 19

Solve: $(x_1 + x_2 + x_3)^3 - x_1 = \alpha_1^3$, $(x_1 + x_2 + x_3)^3 - x_2 = \alpha_2^3$, $(x_1 + x_2 + x_3)^3 - x_3 = \alpha_3^3$.

Either the Greek text has been corrupted, or Diophantus has not expressed the means by which he obtained his solution. Bachet believes that he was aided by luck, which I could hardly admit, since I believe that his method is not difficult to rediscover.

It is a question of finding a square greater than 2, but smaller than 3, whose difference from 3 can be divided into three cubes.

Let us take as the root for the sought-for square, an expression composed of a term of x and -1, for example $x - 1$. If I subtract the square of this expression from 3 , there remains: $2 + 2x - x^2$, which is a matter of decomposing into a sum of three cubes in such a way that the equation will reduce to two terms of consecutive degree.

The solution can be arrived at in an infinite number of ways: let $1 - \frac{1}{3}x$ be the root of one of the cubes; for that of the second, we take $1+x$, in order that the sum of these two cubes gives $2x$ for the term of the first degree; the root of the third must include only one term in x, which must moreover have the sign $-$, in order that the value of x remain within the assigned limits. But it will not be difficult to choose the coefficient of this term in x in such a way that the solution effectively falls within the limits of the question.

This done, it is clear that our first cube will be smaller than unity, as we wish it to be. On the contrary, the second is greater than unity, and the third has the sign $-$. It follows that we must find two cubes whose sum is equal to the difference between the second and the third; we thus arrive, as Diophantus does, to his second operation.

"But we have," he says, "in the Porisms, that the difference between two arbitrary cubes is also the sum of two cubes."

Here Bachet is stuck, and, since the Porisms of Diophantus are missing, he asserts that this is a problem which is not possible except under a certain condition. In fact, he teaches how to divide the difference between two cubes into two cubes, but only when the greater of the given cubes is more than twice the smaller, and he frankly confesses that he does not know the general way of dividing the difference between two arbitrary cubes into two other cubes. I have shown above, in regards to problem IV, 2, the general solution to this question, and to others relative to the same subject.

29. – Diophantus, V, 24

> Find three squares such that the product of the three, plus one chosen arbitrarily among them, makes a square. The problem is reduced to finding three right triangles such that the ratio of the product of the bases to the product of the heights is a square.

Here is how I restore and explain the method of Diophantus, which was not understood by Bachet.

Having taken 3, 4, 5 as the first triangle, for which the product of the sides adjoining the right angle is 12, Diophantus says, "we are brought to seeking out two triangles such that the product of the sides adjoining the right angle of one of them is 12 times the product of the sides next to the right angle of the other." The reason for this is that if we multiply the two products by each other, we will have a planar number similar to 12, and that then, by multiplying this last number by 12, we will have a square, which is what the proposed problem demands.

Diophantus continues: "Yet the area of one of these triangles will be 12 times that of the other," which is self-evident. "But instead of 12 times, we could say 3 times." Indeed, since 3 is the quotient of 12 divided by the square 4, the general multiplication of bases by heights will always give a square, since, if we divide a square by a square, the quotient is still a square.

What follows in the text of Diophantus does not give the sought-after solution, but I will restore it as follows:

In the proposed case, we will form one of the two triangles from the numbers 7 and 2, the other from the numbers 5 and 2. The first triangle will have its area triple that of the second, and this pair of triangles will satisfy the question.

For the rest, to find two right triangles whose area is in a given ratio, here is the general rule.

Let $\frac{r}{s}$ be the given ratio, assuming $r > s$. We will form the greater triangle from the numbers $2r+s$ and $r-s$, and the smaller triangle with the numbers $r+2s$ and $r-s$.

We can also form the two triangles in the following manners:

The first from $2r-s$ and $r+s$, and the second from $2s-r$ and $r+s$;

The first from $6r$ and $2r-s$, and the second from $4r+s$ and $4r-2s$;

The first from $r+4s$ and $2r-4s$ and the second from $6s$ and $r-2s$.

From the preceding, we can deduce a method to find three right triangles whose areas are proportional to three given numbers, provided that the sum of two of these numbers be four times the third.

For example, let the numbers r, s, t be given, and let us assume that $r+t=4s$. We will form the three triangles as follows: the first from $r+4s$ and $2r-4s$, the second from $6s$ and $r-2s$, and the third from $4s+t$ and $4s-2t$. (I have supposed that $r > t$.)

We could just as well derive a means of finding three right triangles in numbers, such that their areas form a right triangle.

In effect, we will bring the question to finding a triangle for which the sum of the base and the hypotenuse be four times its height. This problem is easy, and the sought-for triangle will be similar to the following: 17, 15, 8.

As for the three triangles, the numbers that generate them will be: for the first, 49 and 2; for the second, 47 and 2; for the third, 48 and 1.

Finally we will also have the means of finding three triangles whose areas are proportional to three given squares, by assuming that the sum of two of these squares be four times the third. In the same way, we will also be able to find three triangles having equal areas. Finally, we can construct right triangles in an infinite number of ways, whose areas will be in a given ratio, by multiplying one of the terms by the ratio or by multiplying both terms by the given squares, etc.

30. – Diophantus, V, 25

> Find three squares, such that the product of the three, minus one taken arbitrarily among them, makes a square. The problem is brought to finding three right triangles such that the ratio of the product of the hypotenuses to the product of the heights be a square.

Just as for the preceding, Bachet has treated this problem by leaving aside the method of Diophantus, which therefore still remains to be brought to light and explained. To this effect, we will find two right triangles such that the product of the hypotenuse times the base in one of these triangles will be in a given ratio with the same product for the other triangle.

This question tormented me for a long time, and whoever will try to solve it will be able to recognize that it is really difficult; I have finally discovered a method for the general solution.

Let there be sought two triangles such that the product of the hypotenuse times the height, in one of these triangles, be double the same product in the other.

Let a and b be the numbers that generate one of the triangles, and a and d those for the other.

For the first, the product of the hypotenuse times the height will be $2ba^3 + 2b^3a$.

For the second, the same product will be $2da^3 + 2d^3a$. It is required that the first of these products be twice the second; consequently:

$$ba^3 + b^3a = 2da^3 + 2d^3a.$$

Dividing all the terms by a,

$$ba^2 + b^3 = 2da^2 + 2d^3.$$

Transposing terms, we have:

$$2d^3 - b^3 = ba^2 - 2da^2.$$

To solve the question, the quotient of $2d^3 - b^3$ divided by $b - 2d$ must be a square.

It follows that the task comes down to finding two numbers, b and d, such that the excess of double the cube of one over the cube of the other gives a square, whether we divide it or multiply it (for it comes to the same thing) by the excess of twice the second over the first.

Let $x + 1$ be one of these numbers, and let the other be 1. The excess of twice the cube of the one over the cube of the second is $1 + 6x + 6x^2 + 2x^3$; the excess of double the second number over the first is $1 - x$. The product of $1 + 6x + 6x^2 + 2x^3$ by $1 - x$ must be a square. Yet this product is $1+5x-4x^3-2x^4$, which can be equated to the square of $1+\frac{5}{2}x-\frac{25}{8}x^2$. The rest offers no more difficulty.

To extend this method to the case of a given ratio, it will suffice to take, for one of the numbers, the sum of x and the excess of the greater term of the ratio over the lesser term; for the other number, this same excess. This is also what we have done for the ratio of 2 to 1. In this way the independent term of x in the final product will indeed be a square,[5] and the equation will be able to be treated easily. Its solution will lead to two numbers representing b and d and we can thus return to the original problem.

By looking again at what I wrote above on this question of Diophantus, I was about to erase everything because in reality it is not this which returns to the problem whose solution I revealed. However, if I have erred in reducing one question to another, this last question is no less valuably solved. My work has then been misplaced rather than lost and I leave what I have written in the margin.

As for the question of Diophantus itself, I have submitted it to a new examination and after employing all the resources of my method, I finally obtained the general solution. However I will only give one example of it, whose numbers will sufficiently demonstrate by themselves that it is not chance, but rather a regular method which permitted me to find them.

[5]Or the term independent of x? — *le terme ind'ependant de x*

Diophantus in fact proposed to find two right triangles, such that the product of the hypotenuse by the height of the first will be to the same product in the second triangle in a ratio of 5 to 1.

Here are two triangles which satisfy the condition:

	First triangle	Second triangle
Hypotenuse	48,543,669,109	42,636,752,938
Base	36,083,779,309	41,990,695,480
Height	32,472,275,580	7,394,200,038

31. – Diophantus, V, 30

Solve $x_1^2 + x_2^2 + a = \square, \quad x_2^2 + x_3^2 + a = \square, \quad x_3^2 + x_1^2 + a = \square.$

Thanks to this problem, we obtain the solution to a question which, otherwise, would appear very difficult: *Given a number, find four other numbers such that their sums taken two at a time, increased by a given number, make squares.*

Let the number 15 be given. We will begin by seeking out, according to the solution of Diophantus, three squares such that their sums taken two at a time, increased by a given number, will make squares. Let 9, $\frac{1}{100}$, $\frac{529}{225}$ be these three squares. We will take, for the first of the four sought numbers: $x^2 - 15$, and for the second: $6x + 9$ (since 9 is one of the found squares, and 6, the coefficient of x, is twice the root of the square). Following the same procedure, we will take $\frac{1}{5}x + \frac{1}{100}$ as the third number and $\frac{46}{15}x + \frac{529}{225}$ as the fourth.

With these positions, we have satisfied the three conditions of the problem. For if we take the sum of the first number with any of the following numbers, and add 15, we will have a square.

It must also be that we arrive at squares when we add 15 to the sum of the second and the third, the third and the fourth, or the second and the fourth. We will thus have a triple equation, which will be easy to deal with, because, thanks to the construction whose technique we have borrowed for the problem of Diophantus, in each of the expressions that we seek to equate to a square, the constant term will be a square, and there will only be a term in x besides this. See what I said on this subject in my remarks on problem VI, 24.

32. – Diophantus, V, 31

Solve $x_1^2 + x_2^2 - a = \square, \quad x_2^2 + x_3^2 - a = \square, \quad x_3^2 + x_1^2 - a = \square.$

An analogous technique to that which we used in the preceding question (to find four numbers such that their sums taken two at a time, increased by a given number, make squares), can serve to pass from this question of Diophantus to the search for four numbers such that their sums taken two at a time, decreased by a given number, make squares.

For the first number, we will take: x^2+ the given number. For the second, we will add the first square we found according to Diophantus to a term in x having twice the root of this square as its coefficient, etc. The rest is obvious.

33. – Diophantus, V, 32

Solve: $x_1^4 + x_2^4 + x_3^4 = \square.$

Why does he not look for two biquadratics whose sum is a square? It is because that problem would be impossible, as our method of demonstration can prove beyond a doubt.

34. – Diophantus, VI, 3

> "Find a right triangle in numbers whose area, increased by a given number, makes a square." – Viète had wrongly assumed, as Bachet remarks, that the given number must be the sum of two squares. In the following problems, the area must be decreased by a given number.

Here you see, without a doubt, the origin of Viète's error: this illustrious savant would have equated the area to the difference between two biquadratics, such as $x^4 - 1$, to make a square, by adding the quintuple of a square to it, with 5 being the given number.

Since this last number is the sum of two squares, we can indeed find a square, whose quintuple, decreased by unity, makes a square. Let us take $x+1$ as the root of this square to quintuple (the coefficient of x could be taken as something other than unity). The quintuple of the square will be $5x^2+10x+5$. By adding the area, $x^4 - 1$, we will have the sum $x^4 + 5x^2 + 10x + 4$, which we will have to equate to a square, which is easy, since the constant term is square, as follows from the hypothesis which has been added as a condition.

But Viète did not see that the problem can just as well be solved by taking as the area, not $x^4 - 1$, but rather $1 - x^4$, for then the question is immediately brought to causing the given number – 5, 6, or another number – multiplied by a square, to make another square, after adding unity. This can be solved very easily and without exception, since unity is a square.

I have solved this question, as well as the two following ones, by a particular method, which permits us – if we are seeking, for example, a triangle whose area, increased by 5 makes a square – to give such a triangle with minimal numbers: $\frac{9}{3}, \frac{40}{3}, \frac{41}{3}$ gives a triangle with area 20, which, when 5 is added, makes 25.

But this is not the place to develop the principle and the use of this method. The margin would not suffice for it, since I would have much to say on this subject.

35. – Diophantus, VI, 6

> Find a right triangle, such that the area, increased by one of the sides adjacent to the right angle, makes a given number.

This problem and the following can be solved otherwise:

For this, if we form a triangle with the given number and unity, and if we divide the sides by the sum of the given number and unity, then the quotients will constitute the sought-for triangle.

36. – Diophantus, VI, 7

> Find a right triangle, such that the area, decreased by one of the sides adjacent to the right angle, makes the given number.

If we form a triangle with the given number and unity, and if we divide the sides by the difference between the given number and unity, then we will have the sought-for triangle.

For the rest, this question is susceptible to an infinite number of solutions, by the procedure which permits us to find an infinite number of double equations of this sort. I have indicated below the use of this procedure, in my remarks on question 24.

What's more, we will similarly have an infinite number of solutions for the four following questions, which was recognized neither by Diophantus nor by Bachet. But why did neither of them add the following problem:

Find a right triangle, such that one of the sides adjacent to the right angle, decreased by the area, makes a given number.

They seem now to have known the solution, because it is not immediately provided by the double equation. However, we can find it easily with our method.

This third case can similarly be added to the following questions.

37. – Diophantus, VI, 8 and 9

> Find a right triangle, such that the area, increased (or decreased) by the sum of the sides adjacent to the right angle, makes a given number.

With our method, we can add the following problem:

Find a right triangle such that the sum of the sides adjacent to the right angle, decreased by the area, makes a given number.

38. – Diophantus, VI, 10 and 11

> Find a right triangle, such that the area, increased (or decreased) by the sum of the hypotenuse and one of the sides, makes a given number.

With our method, we can add the following problem:

Find a right triangle such that the sum of the hypotenuse and one of the sides adjacent to the right angle, decreased by the area, makes a given number.

Similarly, we will add the following to the commentaries of Bachet:

Find a right triangle such that the hypotenuse, decreased by the area, makes a given number.

39. – Diophantus, VI, 13

> Find a right triangle, such that the area, increased by one or the other of the sides adjacent to the right angle, makes a square in both cases.

Diophantus only gives triangles of a single type as a solution to the problem. Our method provides an infinite number of triangles of different types, which are derived successively from the solution of Diophantus.

Indeed, let the 3, 4, 5 triangle already be found, which satisfies this condition "that the product of the two sides adjacent to the right angle makes a square, if we add to it the product of the greater of these sides by the difference between them and by the area of the triangle." It comes down to finding another which enjoys the same property.

Let 4 be the greater side adjacent to the right angle of the sought-after triangle and let $3 + x$ be the smaller. The product of the two sides adjacent to the right angle, if *on lui ajoute le produit du plus grand des deux côtés par leur différence et par l'aire du triangle* will make $36 - 12x - 8x^2$, which expression must be made equal to a square. On the other hand, since the sides 4 and $3+x$ are those next to the right angle of a right triangle, the sum of their squares must be a square. Now this sum is $25 + 6x + x^2$, the second expression which must be made equal to a square.

We have therefore a double equation, which is easy to solve:

$$36 - 12x - 8x^2 = \square, \quad 25 + 6x + x^2 = \square.$$

40. – Diophantus, VI, 14

> Find a right triangle, such that the area, decreased by one of the other of the two sides adjacent to the right angle, makes a square in either case.

With our method, we will be able to solve the following question which, otherwise, is very difficult:

Find a right triangle such that both of its sides, decreased by the area, make a square.

41. – Diophantus, VI, 15 and 17

> Find a right triangle such that its area, decreased (or increased) by either the hypotenuse or one of the two sides, makes a square.

We can, with our method, take on the following question, which, otherwise, is very difficult:

Find a right triangle such that by removing from the area, either the hypotenuse or one of the sides, we will have a square.

42. – Diophantus, VI, 19

> Find a right triangle, whose perimeter is a cube, and makes a square from the sum of its area and the hypotenuse.

Can there be, in whole numbers, another square besides 25 which, increased by 2, makes a cube? This certainly appears difficult to discuss at first; however, I can prove, by a rigorous demonstration, that 25 is indeed the only whole-number square which is two less than a cube. In fractional numbers, the method of Bachet provides an infinite number of such squares, but the theory of whole numbers, which is very beautiful and quite subtle, has not been known before now, either by Bachet, or by any other author whose works I have seen.

43. – Commentary of Bachet on Diophantus, VI, 24

This commentary is dedicated to the theory of the *double equation*.

Where *double equations* or $\delta\iota\pi\lambda o\iota\sigma\acute{o}\tau\eta\tau\epsilon\varsigma$ do not suffice, recourse must be had to *triple equations* or $\tau\rho\iota\pi\lambda o\iota\sigma\acute{o}\tau\eta\tau\epsilon\varsigma$, a discovery that I have made which leads to the solution to a great mass of very beautiful problems.

For example, let us seek to equate the squares of the expressions

$$x + 4, \quad 2x + 4, \quad 5x + 4.$$

Here is a *triple equation* which is easy to resolve by use of a *double equation.*

If, in fact, we substitute for x an expression which, increased by 4, makes a square, $x^2 + 4x$ for example, the three above expressions which we seek to equate to squares will become

$$x^2 + 4x + 4, \quad 2x^2 + 8x + 4, \quad 5x^2 + 20x + 4.$$

The first is a square by construction. Therefore, these conditions remain to be satisfied:

$$2x^2 + 8x + 4 = \square, \quad 5x^2 + 20x + 4 = \square,$$

a *double equation*, which, in reality, has only a single unique solution, but from this solution we will be able to derive another, from this second a third, and so on indefinitely.

To this end, when we will have found a value for x, we will substitute for x the binomial formed from x plus the value which we have just obtained. This procedure will produce an infinite number of solutions each derived from the preceding one, and being added to the end of the list.

It is thanks to this invention that we can give an infinite number of triangles of the same area, which Diophantus seems not have known how to do, since he resorts to his problem V, 8, where he seeks out only three triangles of the same area to solve the following problem with three unknowns. But this last question, after the discovery I have made, can be extended to an indefinite number of unknowns.

44. – Same Commentary

To this treatise on *double equations,* we could make numerous additions on points unknown to both the ancients and the moderns. But it will be enough to establish the importance of our method and to demonstrate its use, to solve here the following question, whose difficulty is undeniable.

Find a right triangle in whole numbers, such that the hypotenuse is a square, as is the sum of the sides adjacent to the right angle.

The sought-after triangle is represented by the three following numbers:

4,687,298,610,289,   4,565,486,027,761,   1,061,652,293,520,

and is created from the two numbers 2,150,905 and 246,792.

I have found, by another method, the solution to this other question:

Find a right triangle in whole numbers, such that the square of the difference between the sides adjacent to the right angle, minus twice the square of the smaller of these sides, makes a square.

The triangle 1525, 1517, 156, formed from the numbers 39 and 2, is one of those which satisfy this question.

Moreover, I add with confidence that the two triangles above are the smallest in whole numbers that satisfy the proposed questions.

Here is my method: find, according to the ordinary procedure, the solution to the proposed question. If, after completing the calculations, the operation leads nowhere, because the value to give to the unknown is affected by the $-$ sign and must be regarded as smaller than zero, I boldly affirm that one need not give up hope and stand dumbfounded, as Viète says, although that is what he did, like the ancient analysts. On the contrary, one must take on the question again by substituting for the unknown the binomial formed of x minus the number found in the first operation as the value affected by the minus sign. One will thus have a new equation which will lead to a solution in true numbers.

It is by this means that I have solved the two questions above; otherwise they are very difficult. Similarly, I have shown that a sum of two cubes can be decomposed into two other cubes, and I have given the construction which could require reiteration of the operation up to three times. It often happens that the sought-after truth obliges the most able and industrious of analysts to restart the calculations several times, as experience quickly teaches.

45. – Problem 20 of Bachet on Diophantus, VI, 26

Bachet. – Find a right triangle whose area is a given number.

The area of a right whole-number triangle cannot be a square.

I will demonstrate this theorem that I have discovered. Moreover, I did not discover it without a painful and laborious meditation, but this type of demonstration will lead to marvelous progress in the science of numbers.

If the area of a triangle were a square, there would be two biquadratics whose difference would be a square. It follows that we would also have two squares whose sum and difference are squares. Consequently, we would have a square number, the sum of a square and double a square, with the condition that the sum of the two squares which make it up, is also a square. But if a square number is the sum of a square and double a square, its root is also the sum of a square and the double of a square, which I can prove without difficulty. We will conclude from this that this root is the sum of the two sides adjacent to the right angle of a right triangle, where one of the squares will form the base, and double the other will make up the height.

This right triangle will then be formed by two square numbers, of which the sum and the difference will also be squares. But we will prove that the sum of these two squares is smaller than that of the first two which we had also assumed to have a sum and a difference that are squares. Therefore, if we give two squares of which the sum and the difference are squares, we also give, in whole numbers, two squares enjoying the same property, and whose sum is less.

By the same reasoning, we will then have another sum, smaller than that deduced at first, and by continuing this process indefinitely we will always find smaller and smaller whole numbers satisfying the same conditions. But this is impossible, since if we are given a whole number, there cannot be an infinite number of whole numbers smaller than it.

The margin is too narrow to receive the complete demonstration with all its developments.

By the same procedure, I have discovered and demonstrated that there is no triangular number besides unity which is biquadratic.

46. – Commentary of Bachet on Diophantus Numb. Polyg. 9

Find a polygon with a given side, and perform the inverse.

I will set out here, without demonstration, a very beautiful and very remarkable proposition that I have discovered:

In the natural progression starting at unity, the product of an arbitrary number times its immediate successor makes double the triangle of the first number. If the multiplier is the triangle of the number immediately following, we have three times the pyramid of the first number. If it is the pyramid of the number immediately following, we have the quadruple of the "*triangulo-triangulaire*" of the first number, and so on indefinitely, following a uniform and general rule.

I deem that a more beautiful or general theorem regarding numbers could not be stated. I have neither the time nor the space to put the demonstration in this margin.

47. – Bachet, Appendix, II, 27

$$1 = 1^3, \quad 3 + 5 = 2^3, \quad 7 + 9 + 11 = 3^3, \quad 13 + 15 + 17 + 19 = 4^3, \quad \ldots$$

Here you have the way that I will state this proposition in a more general manner:

In any polygonal progression, unity constitutes the first *column*,[6] the sum of the two following numbers, decreased by the first triangle multiplied by the excess above 4 or the number of angles of the polygon, will form the second *column.* The sum of the three following numbers, decreased by the second triangular number multiplied by the excess above 4 of the number of angles in the polygon, will form the third *column,* and so on indefinitely, following the same law.[7]

48. – Bachet, Appendix, II, 31

[6] *colonne*

[7] There is a lengthy footnote inserted here by the editors of the Œuvres, which will be translated later.

$$a^3[\frac{n(n+1)}{2}]^2 = \Sigma_1^n (na)^3.$$

It follows from this that the product of the cube of the greatest number $[(na)^3]$ times the number of the terms $[n]$ is smaller than the quadruple of the sum of the cubes $[\Sigma_1^n (na)^3]$.