Pascal to Fermat

Wednesday, July 29, 1654

Sir,

1. Restlessness affects me just as it does you, and although I am already in bed, I cannot restrain myself from telling you that last night I received from M. de Carcavi your letter on parts, which impressed me more than I can put into words. I have not had the leisure to extend myself, but, in a word, you have found the two parts[1] of dice and players in perfect justice: I am completely satisfied with your treatment, since I no longer doubt that I am right, after the admirable encounter where I find myself in agreement with you.[2]

I admire the method of parts still more than that of dice; I have seen several people who found that of dice, such as M. le Chevalier de Méré, who had posed these questions to me, and also M. de Roberval: but M. de Méré found neither the fair value of the parts, nor the means of determining it, and so I found myself the only one who knew this proportion.

2. Your method is quite certain, and is that which first came to my mind in this study; but, because the pain of combinations is excessive, I found a short-cut and indeed another method, much shorter and more elegant, that I would like to be able to state here briefly: for from now on I would like to open my heart to you if I may, such has been the joy of seeing that we are in agreement. I clearly see that the truth is the same in Toulouse as in Paris.

[1] *Part* signifies here the reparation to made among players, according to their relative chances of success, if the game is abandoned before it ends.

[2] Je ne doute plus maintenant que je ne sois dans la vérité, après la rencontre admirable où je me trouve avec vous.

This is how I determine the value of each of the parts, when two players play, for example, for *three* wins, and each has put 32 pistoles into play:

Let us say that the first has won *two* and the other has won *one*; they now play one round, such that if the first wins, he takes all the money in play, namely 64 pistoles; if the other wins, there are *two* parts for *two* parties, and consequently, if they wish to separate, each would take their wager, namely 32 pistoles.

Therefore consider, Sir, that if the first wins, 64 belong to him; if he loses, 32. Therefore if they do not wish to play another round and wish to separate without playing, the first may say, "I am sure to have 32 pistoles, since even losing would leave me with that many, but for the other 32, perhaps I will win them, perhaps you will – the chance is equal. Therefore let us split these 32 pistoles in half, and I will take the 32 which are certainly mine." He would then have 48 pistoles, and the other would have 16.

Now let us say that the first has won *two* rounds and the other has won *none*, and they begin to play a round. The stakes are that if the first wins, he takes all the money – the 64 pistoles – and if the other wins, then they return to the preceding case, where the first has won *two* and the other *one*.

Now, we have already shown that in that case there belongs to the winner of *two* rounds, 48 pistoles: therefore, if they do not wish to play this round, he must say: "If I win, I will win all, namely 64: if I lose, I will still legitimately have 48: therefore give me the 48 that are certainly mine even if I lose, and let us divide the remaining 16 evenly, since we have the same odds of winning." Thus he will have 48 and 8, which make 56 pistoles.

Finally, let us say that the first has won only *one*, and the other *none*. You see, sir, that, if they begin to play a new round, where the stakes are that if the first wins, he will be *two* to *none*, and then would have 56 by the preceding. If he loses, they are even, leaving him with 32 pistoles. Therefore he must say: "If you do not wish to play, give me the 32 pistoles which are certainly mine, and let us split the rest of the 56 in half. From 56 remove 32, leaving 24. Therefore split the 24 in half, taking 12 for you and 12 for me, which makes 44 with my 32."

Now by this means you see by simple subtractions, that in the first round, 12 pistoles were at stake; 12 in the second; and 8 for the last.[3]

[3] That is, after the first round (described last by Fermat), the player went from 32 to 44 pistoles, an increase of 12. In the second round, the winnings increased another 12 to make 56. In the third and last round (where the other player had won once), he was down to 48, a loss of 8.

Now, to remove the mystery, since you see well enough at first glance, I write this only to be certain that I am not in error, the value (by value, I intend only the money of the opponent) of the latter part of *two* is double the latter part of *three* and four times the latter part of *four* and eight times the latter part of *five*, etc.

3. But the proportion of the first parts is not so easy to find: therefore it is thus, for I wish to hide nothing, and here is the problem for which I have worked out many cases, since I very much enjoy it:

Given as many rounds to be won as you wish, find the value of the first.

For example, let the number of rounds be eight. Take the first *eight* even numbers, and the first *eight* odd numbers, to wit:

$$2, 4, 6, 8, 10, 12, 14, 16$$

and

$$1, 3, 5, 7, 9, 11, 13, 15.$$

Multiply the even numbers in this manner: the first by the second, the product by the third, that product times the fourth, etc.; multiply the odd numbers in the same way: the first by the second, the product times the third, etc.

The final product of the evens is the *denominator* and the final product of the odds is the *numerator* of the fraction which expresses the value of the first of *eight* rounds: which is to say that if each bets the number of pistoles expressed by the product of pairs, he gets an amount of the other's money expressed by the product of odds.

This can be demonstrated, but with much pain, by combinations such as you have imagined, and I was not able to demonstrate it by this other path that I have just told you, but only by means of combinations. Here are the propositions which lead to it, which are properly considered to be arithmetic propositions concerning combinations, for which I have found the beautiful properties:

4. If out of an arbitrary number of letters, eight for example:

$$A, B, C, D, E, F, G, H$$

you take all the possible combinations of four letters and then all the possible combinations of 5 letters, and then of 6, 7, and of 8, etc., and if you thus take all the possible combinations from the number which is half the total,[4] I say that if you join together the half of the combination of 4 with each of the superior combinations, the sum will be the 4^{th} number of the quaternary procession beginning with the binary, which is half of the multitude.

For example, and I will write it in Latin, since French is worthless:

Let there be an arbitrary number of letters, eight for example:

A, B, C, D, E, F, G, H;

from them, form all the combinations of four-to-four, five-to-five, six-to-six, up to eight-to-eight. I say that, if we add half of the number of combinations of four (namely 35, half of 70), the number of combinations of 5 (which is 56), the number of combinations of 6 (which is 28), the number of combinations of 7 (namely, 8), and finally the combinations of 8 (i.e., 1), we will have the fourth term of the geometric procession starting at 2 and having 4 as its ratio. I say the fourth, since 4 is half of eight.

The terms of this geometric progression, starting at 2 and having 4 as the ratio, are:

2, 8, 32, 128, 512, etc.

The first is 2, the second 8, the third 32, and the fourth is 128; now:

128 equals

+ 35, half the combinations of 4 letters
+ 56 combinations of 5 letters
+ 28 combinations of 6 letters
+ 8 combinations of 7 letters
+ 1 combination of 8 letters.[5]

5. There you have the first purely arithmetic proposition: the next regards the doctrine of parts and is:

It must first be said: if one has won *one* round of 5, for example, and thus needs 4 more, the game will undoubtedly be determined after 8 rounds, which is the double of 4.

[4]For example, we start at combinations of 4 letters when the total is 8.

[5]Pascal now switches back to French.

The value of the opponent's money after the first of five rounds is that fraction which has as its numerator half the combination of 4 out of 8 (I take 4 because it is equal to the number of remaining rounds to win, and 8 because it is twice 4), and as its denominator this same numerator plus all the superior combinations.

Thus, if I have *one* round out of five, then $\frac{35}{128}$ of my opponent's money belongs to me; that is to say that if he wagered 128 pistoles, I will take 35 and leave him the remainder, 93.

Yet this fraction $\frac{35}{128}$ is the same as this one: $\frac{105}{384}$, which is made by the product of evens in the denominator and the product of odds in the numerator.

Without doubt this will all be very clear to you, if you give it even the least attention: which is why I see no use in continuing to explain this to you.

6. Nonetheless I send you one of my old Tables; I haven't the leisure to copy it, so I will remake one for myself.

Here you see how at all times the value of the first round is equal to that of the second, which is easily found by combinations.

Similarly, you see that the number of the first line always increases; the same for the second line; the same for the third.

But following that, the number of the fourth line decreases, that of the fith, etc. This is strange.

Si on joue chacun 256 en

Il m'appartient, sur les 256 pistoles de mon joueur, pour la	6 parties.	5 parties.	4 parties.	3 parties.	2 parties.	1 partie.
1re partie	63	70	80	96	128	256
2e partie	63	70	80	96	128	
3e partie	56	60	64	64		
4e partie	42	40	32			
5e partie	24	16				
6e partie	8					

(Translation coming)

Il m'appartient, sur les 256 de mon joueur, pour	Si on joue 256 chacun en					
	6 parties.	5 parties.	4 parties.	3 parties.	2 parties.	1 partie.
la 1re partie	63	70	80	96	128	256
les 2 premières parties	126	140	160	192	256	
les 3 premières parties	182	200	224	256		
les 4 premières parties	224	240	256			
les 5 premières parties	248	256				
les 6 premières parties	256					

(Translation coming)

7. I do not have the time to send you the demonstration of a difficulty which strongly astonished M. de Méré, for he has a very good mind, but he is not a geometer (which, as you know, is a great defect) and does not even understand that a mathematical line may be infinitely divided and strongly believes that it is composed of a finite number of points, and I have never been able to shake him from this conviction. If you are able to, he can be perfected.

He told me therefore that he had found falsity in the numbers by this reasoning:

If one undertakes to roll a *six* with a die, there is an advantage in undertaking to do it in four rolls: 671 to 625.

If you try to roll double-six[6] with two dice, there is a disadvantage in trying to do it in 24 rolls.

And nevertheless 24 is to 36 (which is the number of faces on two dice) as 4 is to 6 (which is the number of faces on one die).

There you have what was the great scandal which made him haughtily say that the propositions are not constant and that Arithmetic in in conflict with itself: but you will easily see the reason by the principles yourself.

I will put in order all that I have done on this account, once I have completed some geometric treatises that I have been working on for some time already.

8. I have also done some works on arithmetic, subjects about which I wish to ask your advice regarding this demonstration.

[6] *sonnés*

I take the lemma that everybody knows: that the sum of as many numbers as one likes of the continuous progression from unity:

1, 2, 3, 4,

being taken twice, is equal to the last, 4, multiplied by the next greater, 5: that is to say that the sum of the numbers up to A, being taken twice, is equal to the product

$$A \text{ times } (A + 1).$$

Now, I come to my proposition:[7]

The difference between two consecutive cubes with one subtracted, is equal to the sixfold sum of the numbers from 1 to the root of the smaller cube, inclusively.[8]

Let R and S be the two roots, which differ by unity; I say that $\mathrm{R}^3 - \mathrm{S}^3 - 1$ is equal to six times the sum of the numbers from 1 to S.

Let $\mathrm{S} = a$, and consequently $\mathrm{R} = a + 1$,

$$\mathrm{R}^3 \text{ or } (a+1)^3 = a^3 + 3a^2 + 3a + 1^3 \text{ and } \mathrm{S}^3 \text{ or } a^3 = a^3.$$

The difference $\mathrm{R}^3 - \mathrm{S}^3 = 3a^2 + 3a + 1^3$, and removing unity,

$$\mathrm{R}^3 - \mathrm{S}^3 - 1 = 3a^2 + 3a.$$

But according to the lemma, the double of the sum of the numbers from 1 to S or a is $a(a+1)$ or $a^2 + a$. Therefore $3a^2 + 3a$ will be six times this sum.

Now $3a^2 + 3a = \mathrm{R}^3 - \mathrm{S}^3 - 1$; therefore $\mathrm{R}^3 - \mathrm{S}^3 - 1$ is six times the sum of the numbers from 1 to S or a.

QED

No one has given me trouble for the preceding, but I have been told that not everyone today is accustomed to this method; for myself, I claim, without giving myself too much credit, that this must be admitted to be an excellent sort of demonstration: nevertheless I await your thoughts on it with all respect.

[7] Pascal switches to Latin here.

[8] e.g., $4^3 - 3^3 = 64 - 27 = 37$, $37 - 1 = 36$, $36 = 6 \times (1 + 2 + 3)$.

All my demonstrations in Arithmetic are of such a nature.

9. Here are still two difficulties:

I have demonstrated a plane proposition by using a cube of one line compared to the cube of the other: I affirm that this is purely geometric and of the greatest purity.

Similarly, I solved the problem:

Of four planes, four points, and four spheres, any four being given, find a sphere which, tangent to the given spheres, passes through the given points and leaves on the planes the portions of spheres corresponding to given angles.

and the following:

Of three circles, three points, three lines, any three being given, find the circle which, tangent to the circles and touching the points, leaves on the lines an arc corresponding to a given angle.

I have resolved these problems *planely*, using only circles and straight lines in the constructions; but in the demonstration, I have used solid loci, parabolas and hyperbolas: I nevertheless claim that since my construction is plane, my solution is plane and must be considered such.

It is quite bad to recognize the honor that you have done me in suffering my conversation and to pester you for so long. I never think to write you only two words, and if I do not tell you the thoughts most in my heart, which is that the more I get to know you, the more I admire and honor you, and if you see the degree to which this is so, you would give a place in your friendship to he who is, Sir, your… etc.

PASCAL.