Fermat to Brûlart de Saint-Martin

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March 31(?), 1643

(From the editors:)

We are without a doubt in the presence of a letter (or rather a writing) of Fermat addressed to Pierre Brûlart, called M. de Saint-Martin in some of Fermat's other letters. As for the date given on the manuscript, we believe that it is erroneous and that it was confused with another letter Fermat wrote to Brûlart, from May 31, 1643 (Vol. 2 of the *Works*, pp. 258-260), which, however, has nothing in common with this letter. In fact, Fermat had already promised Mersenne to "Satisfy the desire of M. de Saint-Martin on the subject of my method of Maxima and Minima" in a letter of February 16, 1643 (Vol. 2, p. 252). The present writing is without a doubt that which Fermat added to his letter to Mersenne of April 7, 1643, writing: "You now have the response that I have made to M. de Brûlart, attached to this letter. I wrote it in haste, as you will see, and that is the reason that I must ask you not to copy it and to make sure that only M. de Brûlart gets his hands on it." In this same letter of April 7, he gives additional clarifications on some points which could have appeared doubtful in the added writing (Vol. 2, pp. 253-254). This letter was thus written shortly before April 7, 1643 and perhaps the date of March 31 is more suitable than May 31.

The recommendations that Fermat made to Mersenne can explain why the writing does not appear in the 1679 edition. This letter merits attention, all the more because contains a development of the method of extremes which aims at a demonstration. In fact, as Fermat wrote in 1658, Carcavi possessed copies of this method "in all the forms, that is to say both with and without demonstration," (Vol. 2, p. 366), but until now, no one knew of a paper including demonstrations. Moreover, the author uses the second derivative here.

(Fermat's Letter:)

My invention of *Maxima and Minima* has only two or three foundations.

First, I assume that this study seeks out a unique point or a unique term, such as when one would like to *divide a line such that the rectangle under the segments be equal to a given area.* We have two points on the line which satisfy the question, but when we look for the greatest of all these rectangles, we have only a single point which satisfies the condition. In the proposed example, it is the point that divides the line into two equal parts. You see why Pappus in his seventh book, always calls this value *maximum, unique, and singular*, and the same for *minima*; the Greek word is $\mu o \nu \alpha \chi \grave{o} \varsigma$, which so shocked Commandino, that he claimed to have no idea of what Pappus could have meant by this term.

It is therefore necessary to find a unique point on either side of which all the terms of the question are either greater or smaller than that produced by the sought point.

We must therefore compare the unique point with those that can be imagined on either side. This cannot be done easily by a single position, because if, for example, we call by A the point that gives us the unique point, we will need to add or subtract another quantity from it to find the relationship between the unique point and those on either side. Therefore, to make the comparison with another point, we can arbitrarily take a point on one side, and call the line which it gives $A+E$; and to make a comparison with a point on the other side, we will call the line that it gives $A-E$: one is made by addition and the other by subtraction. Then it is necessary to find a method by means of which $A+E$ and $A-E$ will give the same term as does A. This is done so that A represents the middle point. Anything to either of the two sides will exceed or fall short, depending on whether we are seeking the smallest or the greatest.

Yet it appears that my method gives the same question by using $A+E$ or $A-E$, which experience and reason will straightaway make apparent. For $A-E$ always gives the same terms as $A+E$, with the sole difference that in odd powers, they have opposite signs, a result which does not change the equation.

It therefore appears that $A+E$ gives the same equation as $A-E$ by my method. But this is not entirely sufficient, for if we had only to find the same equation by $A+E$ as by $A-E$, we could just as well take the two terms which have, for example, E^2 or E^3, etc., instead of those which have

E simply, and equate them to each other, which would, however, not work.[1] It is therefore necessary, in addition to the preceding precaution, which requires that $A+E$ give the same equation as $A-E$, to add another condition: namely that if $A+E$ gives less than A, then $A-E$ must also give less than A, and similarly, if $A+E$ gives a greater value than does A, then $A-E$ must also give a greater value than A.

I explain, using the following example: *Let us divide a line, such that the solid under one of its segments and the square of the other segment is the greatest.*[2]

Let A be the segment of the line which gives the unique point. If the length of the line be B, then the solid will be: $BA^2 - A^3$.

$A+E$ will give:

$$BA^2 - A^3 + BE^2 - 3AE^2 + 2BAE - 3A^2E - E^3;$$

$A-E$ will give:

$$BA^2 - A^3 + BE^2 - 3AE^2 - 2BAE + 3A^2E + E^3.$$

If we take the terms in which E appears simply, we will have in the two equations of $A+E$ and $A-E$ the same equation, for $2BAE$ will be equated with $3A^2E$ in both of them.[3] If we took the terms which contain E^2, we would have the same equation from $A+E$ and $A-E$, for BE^2 will be equated with $3AE^2$ both of them. Therefore the reason must be given why we take E simply, rather than any of its powers.

It is necessary that in each pair of homogeneous positions, which are compared with $BA^2 - A^3$, each one is less than $BA^2 - A^3$. Therefore

$$BA^2 - A^3 + BE^2 - 3AE^2 + 2BAE - 3A^2E - E^3$$

must be less than

$$BA^2 - A^3,$$

[1]Fermat assumes a knowledge of his *Method* here. Briefly, he determines the value when the variable is A, *ad-equates* it to the value when it is $A+E$ (or $A-E$), and removes common terms from both sides. Then he keeps only the terms containing E simply and equates them, removing the E by dividing it from both sides. See the *Method* and the accompanying animations.

[2]To maximize the product of one segment times the square of the other.

[3]This is by setting the expression with $A+E$ or $A-E$ equal to the expression with A and removing common terms.

and

$$BA^2 - A^3 + BE^2 - 3AE^2 - 2BAE + 3A^2E + E^3$$

must also be less than

$$BA^2 - A^3,$$

which could only happen if the terms containing the lowest power of E (here, simply E) were equated with each other. The reason for this is that the terms containing the lowest power of E always have a greater ratio between them than those measured by E^2 or by E^3, etc., and those which contain E^2 have a greater ratio between them than those measured by E^3, E^4, etc. In this example, if we take $A + E$ and equate the two terms containing only E simply, we will have $2BAE$ on one side and $3A^2E$ on the other; yet the ratio of $2BAE$ to $3A^2E$ is greater than BE^2 to $3AE^2$ (taking the two terms containing E^2). The reason for this is that analytic multiplication doubles B in the former equation, but it has no coefficient in the latter. Therefore, if we equate $2BAE$ with $3A^2E$, then BE^2 will be less than $3AE^2$.

From this, we will prove that all the terms which will be marked by the $+$ sign, will be less than those which will be marked by the $-$ sign. And the greatest power of E, which is always found alone, and which is E^3 here, will not change the order of the equation by the sign that it has, which is clear to us from simple inspection. The principal reason for this is that the two terms containing E^2, having a greater ratio than those terms containing powers higher than E^2, will serve as the key for determining the greatest or the smallest. For if the term marked by $+$ is less than the term marked by $-$, then in this case the proposition is seeking to find the maximum; and if the term marked by $+$ is greater than the term marked by $-$, then it is a question of finding the minimum. For if we use $A - E$, the two terms containing E^2 will both have the same sign [as if we had used $A + E$].[4]

And then all the terms which will be similarly marked by the sign $+$ will be less than those marked by the $-$ sign. Both the method and the reasons that I have put forward, are general.

[4]Here you have the consideration of what Leibniz would call a second differential. When it is positive, the extreme value is a minimum, and when it is negative, the value is a maximum. This demonstrates that Fermat *did* have a means of distinguishing between Maxima and Minima (which some academics have complained was a fault of his method).